14th Philippine Mathematical Olympiad
National Stage
28 January 2012

General Instruction: Solve all problems clearly, legibly and completely.

1. A computer generates even integers half of the time and another computer generates even integers a third of the time. If a_i and b_i are the integers generated by the computers, respectively, at time i, show that the probability that $a_1b_1 + \ldots + a_kb_k$ is even is $\dfrac{1}{2} + \dfrac{1}{2 \cdot 3^k}$.

2. Let f be a polynomial function with integer coefficients and p be a prime number. Suppose there are at least four distinct integers satisfying $f(x) = p$. Show that f does not have integer zeros.

3. If $ab > 0$ and $0 < x < \dfrac{\pi}{2}$, prove that

$$\left(1 + \frac{a^2}{\sin x}\right)\left(1 + \frac{b^2}{\cos x}\right) \geq \frac{(1 + \sqrt{2}ab)^2 \sin 2x}{2}.$$

4. Let $\dagger$ be an operation defined in the set of nonnegative integers with the following properties: for any nonnegative integers x and y,

   (i) $(x+1)\dagger 0 = (0\dagger x) + 1$

   (ii) $0\dagger(y+1) = (y\dagger 0) + 1$

   (iii) $(x+1)\dagger(y+1) = (x\dagger y) + 1$.

   If $123\dagger 456 = 789$, find $246\dagger 135$.

5. There are exactly 120 Twitter subscribers from National Science High School. Statistics show that each of 10 given celebrities has at least 85 followers from National Science High School. Prove that there must be two students such that each of the 10 celebrities is being followed in Twitter by at least one of these students.