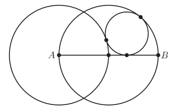
16^{th} Philippine Mathematical Olympiad
Area Stage
7 December 2013

Part I. No solution is needed. All answers must be in simplest form. Each correct answer is worth three points.

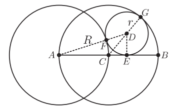
1. 74
2. 18
3. 2021
4. 162
5. $\frac{9}{4} = 2.25$
6. $(m, n) = (1, 12), (2, 6), (3, 4)$
7. 23
8. $5 \leq p \leq 8$
9. $\frac{1}{4} < x < \frac{1}{2}$
10. 72
11. $r = \frac{1}{2}, -2$
12. $\frac{\pi}{4}$
13. 625
14. $\pm 1, 2\sqrt{2}$
15. 34
16. $\frac{2}{5} = 0.4$
17. 91
18. 07
19. $x = \frac{\pi}{3}, \frac{2\pi}{3}$
20. 11.2 cm

Part II. Show the solution to each item. Each complete and correct solution is worth ten points.

1. Two circles of radius 12 have their centers on each other. As shown in the figure, A is the center of the left circle, and AB is a diameter of the right circle. A smaller circle is constructed tangent to AB and the two given circles, internally to the right circle and externally to the left circle, as shown. Find the radius of the smaller circle.



Solution:



Let R be the common radius of the larger circles, and r that of the small circle. Let C and D be the centers of the right large circle and the small circle, respectively. Let E, F and G be the points of tangency of the small circle with AB, the left large circle, and the right large circle, respectively. Since the centers of tangent circles are collinear with the point of tangency, then A-F-D and C-D-G are collinear.

From $\triangle AED$, $AE^2 = (R+r)^2 - r^2 = R^2 + 2Rr$. Therefore, $CE = AE - R = \sqrt{R^2 + 2Rr} - R$.

From $\triangle CED$, $CE^2 = (R-r)^2 - r^2 = R^2 - 2Rr$.

Therefore, $\sqrt{R^2 + 2Rr} - R = \sqrt{R^2 - 2Rr}$. Solving this for r yields $r = \frac{\sqrt{3}}{4}R$. With $R = 12$, we get $r = 3\sqrt{3}$.

2. Let a, b and c be positive integers such that $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c}$ is a rational number. Show that $\dfrac{a^2+b^2+c^2}{a+b+c}$ and $\dfrac{a^3-2b^3+c^3}{a+b+c}$ are both integers.

Solution:

By rationalizing the denominator, $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c} = \dfrac{2013ab - bc + \sqrt{2013}(b^2 - ac)}{2013b^2 - c^2}$. Since this is rational, then $b^2 - ac = 0$. Consequently,

$$\begin{aligned} a^2 + b^2 + c^2 &= a^2 + ac + c^2 = (a+c)^2 - ac = (a+c)^2 - b^2 \\ &= (a-b+c)(a+b+c) \end{aligned}$$

and

$$\begin{aligned} a^3 - 2b^3 + c^3 &= a^3 + b^3 + c^3 - 3b^3 = a^3 + b^3 + c^3 - 3abc \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \end{aligned}$$

Therefore,

$$\frac{a^2+b^2+c^2}{a+b+c} = a - b + c \qquad \text{and} \qquad \frac{a^3-2b^3+c^3}{a+b+c} = a^2+b^2+c^2-ab-bc-ca$$

are integers.

3. If p is a real constant such that the roots of the equation $x^3 - 6px^2 + 5px + 88 = 0$ form an arithmetic sequence, find p.

Solution: Let the roots be $b-d$, b and $b+d$. From Vieta's formulas,

$$-88 = (b-d)b(b+d) = b(b^2-d^2) \quad (1)$$
$$5p = (b-d)b + b(b+d) + (b+d)(b-d) = 3b^2 - d^2 \quad (2)$$
$$6p = (b-d) + b + (b+d) = 3b \quad (3)$$

From (3), $b = 2p$. Using this on (1) and (2) yields $-44 = p(4p^2 - d^2)$ and $5p = 12p^2 - d^2$. By solving each equation for d^2 and equating the resulting expressions, we get $4p^2 + \frac{44}{p} = 12p^2 - 5p$. This is equivalent to $8p^3 - 5p^2 - 44 = 0$. Since $8p^3 - 5p^2 - 44 = (p-2)(8p^2 + 11p + 22)$, and the second factor has negative discriminant, we only have $p = 2$.