EASY 15 seconds, 2 points

1. If $g(x) = \dfrac{x-2}{x}$ and $f(-g(-x)) = \dfrac{x-2}{2x-6}$, find $f(3)$.

   Answer: $\dfrac{5}{14}$

   Solution: Note that $-\dfrac{-x-2}{-x} = 3 \Longrightarrow x = -\dfrac{1}{2}$. Hence $f(3) = \dfrac{-\frac{1}{2}-2}{2\left(-\frac{1}{2}\right)-6} = \dfrac{5}{14}$.

2. A circle with radius 5 is tangent to the x-axis, the y-axis, and the line $4x - 3y + 10 = 0$. Find its center.

   Answer: $(-5, 5)$

   Solution: The x and y-intercepts of the given line are $-5/2$ and $10/3$, respectively. This means that the line and the coordinate axes determine a circle on the second quadrant, and so the center is at $(-5, 5)$.

3. Evaluate: $6 + \cfrac{16}{6 + \cfrac{16}{6 + \cfrac{16}{6 + \cdots}}}$

   Answer: 8

   Solution: Let x be equal to the continued fraction to be evaluated. Assuming the continued fraction converges, we must have $x = 6 + \frac{16}{x}$, which has 8 and -2 as solutions. Since x must be positive, $x = 8$.

4. Simplify $\sqrt{13 + 4\sqrt{3}} + \sqrt{13 - 4\sqrt{3}}$.

   Answer: $4\sqrt{3}$

   Solution: Let $x = \sqrt{13 + 4\sqrt{3}} + \sqrt{13 - 4\sqrt{3}}$; then $x^2 = 48$. Since x is clearly positive, we take $x = \sqrt{48} = 4\sqrt{3}$.

5. Suppose a, b, and c are the sides of a triangle opposite angles α, β, and γ respectively. If $\cos\beta < 0$ and $\cos\alpha < \cos\gamma$, arrange a, b, and c in increasing order.

   Answer: c, a, b

   Solution: β must be obtuse and therefore the largest angle, and so b is the longest side. As for a and c, since α and γ must both be acute, cos is decreasing and thus $\alpha > \gamma$, so $a > c$.

6. How many ordered 5-tuples (a, b, c, d, e) of integers satisfy $10 < a < b < c < d < e < 20$?

   Answer: 126

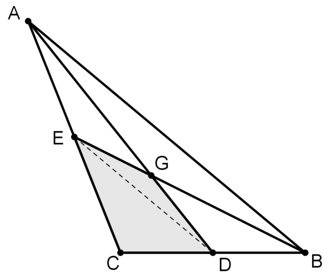
   Solution: We simply need to choose which five of the integers $11, 12, \ldots, 19$ will comprise the 5-tuple. Once the integers are chosen, there is only one way to assign them as a, b, c, d, e to satisfy the given inequalities. Thus, there are $\binom{9}{5} = 126$ possible 5-tuples.

7. In triangle ABC, the medians AD and BE meet at the centroid G. Determine the ratio of the area of quadrilateral $CDGE$ to the area of triangle ABC.

   Answer: $\dfrac{1}{3}$

Solution:
Refer to the figure on the right.

$$\begin{aligned}[CDGE] &= [CDE] + [GED] \\ &= \frac{1}{4}[ABC] + \frac{1}{3}[BED] \\ &= \frac{1}{4}[ABC] + \frac{1}{3}\left(\frac{1}{4}[ABC]\right) \\ &= \frac{1}{4}[ABC] + \frac{1}{12}[ABC] \\ &= \frac{1}{3}[ABC]\end{aligned}$$



8. Find the largest positive integer x such that 2017! is divisible by 19^x.

   Answer: 111

   Solution: There are $\lfloor \frac{2017}{19} \rfloor = 106$ numbers from 1 to 2017 which are divisible by 2017. Among these, five numbers, $19^2, 2 \cdot 19^2, 3 \cdot 19^2, 4 \cdot 19^2, 5 \cdot 19^2$, are divisible by 19^2. Therefore, $19^{111} | 2017!$ and so the largest possible x is 111.

9. Gari is seated in a jeep, and at the moment, has one 10-peso coin, two 5-peso coins, and six 1-peso coins in his pocket. If he picks four coins at random from his pocket, what is the probability that these will be enough to pay for his jeepney fare of 8 pesos?

   Answer: $\dfrac{37}{42}$

   Solution: The only way that Gari will be unable to pay for the fare is if all four coins are 1-peso coins. This has probability $\frac{\binom{6}{4}}{\binom{9}{4}} = \dfrac{5}{42}$, so there is a $\dfrac{37}{42}$ chance that it will be enough to pay for his fare.

10. Solve the inequality $\log_7 \left(\dfrac{1}{x^2 - 3} \right) \geq 0$.

    Answer: $[-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

    Solution: We need all values of x such that $x^2 - 3 > 0$ and $\dfrac{1}{x^2 - 3} \geq 1$. The first has solution $(-\infty, -\sqrt{3}) \cup (3, \infty)$, while the second has solution $[-2, 2]$. The intersection of these two sets is $[-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$.

11. Let $\{a_n\}$ be a sequence such that $a_1 = 20, a_2 = 17$ and $a_n + 2a_{n-2} = 3a_{n-1}$. Determine the value of $a_{2017} - a_{2016}$.

    Answer: -3×2^{2015}

    Solution: We have $a_n = 3a_{n-1} - 2a_{n-2} \Longrightarrow a_n - a_{n-1} = 2(a_{n-1} - a_{n-2})$. Repeated use of this recurrence relation gives $a_{2017} - a_{2016} = 2^{2015}(a_2 - a_1) = -3 \cdot 2^{2015}$.

12. Let A, B, C be positive integers such that the number $1212017ABC$ is divisible by 45. Find the difference between the largest and the smallest possible values of the two-digit number AB.

    Answer: 85

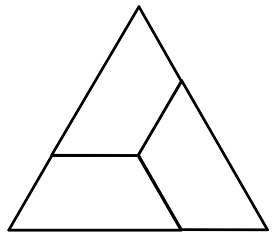
    Solution: Since the number is divisible by 45, it must be divisible by both 9 and 5. If C is 0, then A and B must have a sum of either 4 or 13. If C is 5, then A and B can either have a sum of 8 or 17. Based from these, the largest and smallest values of AB are 98 and 13, respectively, which have a difference of 85.

13. How many ways are there to arrange four 3s and two 5s into a six-digit number divisible by 11?

    Answer: 9

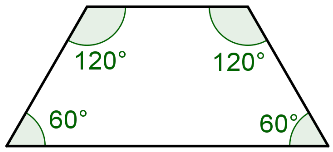
    Solution: From the divisibility rule for 11, we know that the difference of the sum of the odd-positioned digits and the even-positioned digits must be equal to a multiple of 11. The only way this can happen here is that if 3s and 5s are equally distributed over odd and even positions, i.e., two 3s and one 5 will have odd positions, and two 3s and one 5 will have even positions. In the odd positions, there are exactly three ways to arrange them, and same for the even positions. Hence, the number of eligible six-digit numbers is $3^2 = 9$. As a check, we can enumerate them all: 333355, 333553, 335533, 355333, 553333, 335335, 353353, 533533, 533335.

14. An equilateral triangle is divided into three congruent trapezoids, as shown in the figure. If the perimeter of each trapezoid is $10 + 5\sqrt{3}$, what is the side length of the triangle?



<u>Answer:</u> $6 + 3\sqrt{3}$ units

<u>Solution:</u> Each of the trapezoids will have the following interior angles.



We therefore have an isosceles trapezoid. By the manner in which the trapezoids are tiled, the shorter base and the two legs will all have the same length; let it be a. The longer base is then equal to $a + \frac{a}{2} + \frac{a}{2} = 2a$. Therefore

$$\begin{aligned} a + a + a + 2a &= 10 + 5\sqrt{3} \\ 5a &= 10 + 5\sqrt{3} \\ a &= 2 + \sqrt{3} \end{aligned}$$

The side length of a triangle is $a + 2a = 3a = 6 + 3\sqrt{3}$.

15. The symbol ÷ is well-known nowadays to indicate division between two numbers. But during the late medieval period, it had a completely different purpose - it was used to mark words or passages which were spurious, corrupt, or doubtful. The actual name for this "division sign" comes from the ancient Greek word for a sharpened stick or pointed pillar. What is it?

<u>Answer:</u> Obelus

AVERAGE 45 seconds, 3 points

1. Find all real solutions of the following nonlinear system:

$$\begin{aligned} x + 4y + 6z &= 16 \\ x + 6y + 12z &= 24 \\ x^2 + 4y^2 + 36z^2 &= 76 \end{aligned}$$

<u>Answer:</u> $(6, 1, 1)$ and $(-\frac{2}{3}, \frac{13}{3}, -\frac{1}{9})$

<u>Solution:</u> Using the first two equations, we can obtain $y = 4 - 3z$ (by eliminating x) and $x = 6z$ (by eliminating y). Substituting to the third equation gives $(6z)^2 + 4(4 - 3z)^2 + 36z^2 = 76$, which simplifies to $9z^2 - 8z - 1 = 0$. This gives $z = -1/9$ or $z = 1$. Substituting back to the expressions for x and y gives the required answers.

2. Let A be the set of all two-digit positive integers n for which the number obtained by erasing its last digit is a divisor of n. How many elements does A have?

   Answer: 32

   Solution: Let $n = 10a + b$. Since a is a divisor of n, we infer a divides b. Any number n that ends in 0 is therefore a solution. Assuming $b \neq 0$, n must be one of the numbers $11, 12, \ldots, 19$, 22, 24, 26, 28, 33, 36, 39, 44, 48, 55, 66, 77, 88, 99. This gives a total of 32 positive integers.

3. In a tournament, 2017 teams participate. To save time, in each round, instead of a one-on-one battle, three teams (randomly chosen) participate; the game for each round is designed so that exactly one of the three teams survives in the tournament (unless of course it happens that only two teams are remaining, in which case a one-on-one battle does ensue). How many battles must there be to declare a champion?

   Answer: 1008

   Solution: After each battle, 2 teams get eliminated. After 1007 battles, $2017 - 2(1007) = 3$ teams remain, who will fight it out for one more battle.

   Alternatively, there are a total of $\lfloor \frac{2017}{3} \rfloor + \lfloor \frac{673}{3} \rfloor + \lfloor \frac{225}{3} \rfloor + \lfloor \frac{75}{3} \rfloor + \lfloor \frac{25}{3} \rfloor + \lfloor \frac{9}{3} \rfloor + \lfloor \frac{3}{3} \rfloor = 672 + 224 + 75 + 25 + 8 + 3 + 1 = 1008$ rounds.

4. Find the area of the region $\{(x, y) \in \mathbb{R}^2 : |x| - |y| \leq 2 \text{ and } |y| \leq 2\}$.

   Answer: 24 sq. units

   Solution: In the first quadrant, the region is a trapezoid with vertices at $(4, 2), (2, 0), (0, 0), (0, 2)$; this trapezoid has area 6. By symmetry, the region has area $4 \times 6 = 24$ square units.

5. Inside a square of side length 1, four quarter-circle arcs are traced with the edges of the square serving as the radii. It is known that these arcs intersect pairwise at four distinct points, which in fact are the vertices of a smaller square. Suppose this process is repeated for the smaller square, and so on and so forth. What is the sum of the areas of all squares formed in this manner?

   Answer: $\dfrac{1 + \sqrt{3}}{2}$.

   Solution: By similarity, we note that the areas of the squares are in geometric progression. Hence, we need only to find out the area of the first smaller square. Note that the diagonal of the smaller square is the overlap of two line segments of length $\frac{\sqrt{3}}{2}$, with a total length of 1. This is going to be $\sqrt{3} - 1$, and so the area of the smaller square is $\frac{1}{2}(\sqrt{3} - 1)^2 = 2 - \sqrt{3}$.

   Thus, the sum of the areas of all squares is given by $\frac{1}{1-(2-\sqrt{3})} = \frac{1+\sqrt{3}}{2}$.

6. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are 3, -2, and 5. Find the value of $a + b + c$.

   Answer: 167

   Solution: The coefficient of x^3 is 0, so by Vieta's formula, the sum of the roots must be 0, and the fourth root must be -6. Therefore, the polynomial factors as $(x - 3)(x + 2)(x - 5)(x + 6)$. Setting $x = 1$ gives $1 + a + b + c = (1 - 3)(1 + 2)(1 - 5)(1 + 6) = 168$, so $a + b + c = 167$.

7. Given $O(0, 0)$ and $Q(1, 2)$, consider the set of points P for which $OP : PQ = 5 : 4$. It is known that this set is a circle. Find its center.

   Answer: $(\frac{25}{9}, \frac{50}{9})$

   Solution: Suppose we have $P(x, y)$. This implies in particular that $4OP = 5PQ$, or $16OP^2 = 25PQ^2$. Hence, we have

   $$16(x^2 + y^2) = 25(x - 1)^2 + 25(y - 2)^2.$$

   This simplifies to

   $$9x^2 - 50x + 25 + 9y^2 - 100y + 100 = 0$$

and, completing squares,

$$\left(3x-\frac{25}{3}\right)^2+\left(3y-\frac{50}{3}\right)^2=\left(\frac{25}{3}\right)^2+\left(\frac{50}{3}\right)^2-25-100=\frac{2000}{9}.$$

Our center is then given by $(\frac{25}{9},\frac{50}{9})$.

8. Find the remainder when $30!-1$ is divided by 930.

   Answer: 29

   Solution: Since 31 is prime, by Wilson's Theorem, we obtain $30!-30\equiv 0 \pmod{31}$. Because $30!-30\equiv 0 \pmod{30}$ and $\gcd(30,31)=1$, we get $30!-30\equiv 0\pmod{930}$. Therefore, $30!-1\equiv 29 \pmod{930}$.

9. Dominic randomly picks between two words MATHEMATICS and MEMES, each with an equal chance of popping up. From his chosen word, he then randomly draws one letter, with the probability of each letter popping up directly proportional to the number of times it occurs in the word. Given that Dominic drew an M, what is the probability that he, in fact, picked MEMES?

   Answer: $\dfrac{11}{16}$

   Solution: The probability of Dominic picking out the letter M from the word MATHEMATICS is $\frac{2}{11}$. On the other hand, the probability of him picking it from the word MEMES is $\frac{2}{5}$. This gives him, unconditionally, a probability of $\frac{1}{2}\left(\frac{2}{11}+\frac{2}{5}\right)=\frac{16}{55}$ of drawing the letter M. Then, the probability of him drawing the word MEMES, then the letter M from it, is $\frac{1}{2}\cdot\frac{2}{5}=\frac{1}{5}$. Hence, by Bayes' theorem, the probability that the word Dominic drew is MEMES is $\frac{\frac{1}{5}}{\frac{16}{55}}=\frac{11}{16}$.

10. Solve the following inequality.

$$\log_{1/2}x-\sqrt{2-\log_4 x}+1\le 0$$

   Answer: $\frac{1}{\sqrt{2}}\le x\le 16$

   Solution: Note that $2-\log_4 x\ge 0 \Longrightarrow 0<x\le 16$. Let $t=\sqrt{2-\log_4 x}$. Then

$$\log_4 x=2-t^2\Longrightarrow \frac{\log_{1/2}x}{\log_{1/2}4}=2-t^2\Longrightarrow \log_{1/2}x=2t^2-4.$$

   Substituting back to the given inequality, we have

$$2t^2-4-t+1\le 0\Longrightarrow 2t^2-t-3\le 0\Longrightarrow -1\le t\le\frac{3}{2}.$$

   Since $t=\sqrt{2-\log_4 x}$, this means that $0\le\sqrt{2-\log_4 x}\le\frac{3}{2}$, which has solution $\frac{1}{\sqrt{2}}\le x\le 16$.

DIFFICULT 90 seconds, 6 points

1. A triangle ABC is to be constructed so that A is at $(3,2)$, B is on the line $y=x$, and C is on the x-axis. Find the minimum possible perimeter of ΔABC.

   Answer: $\sqrt{26}$

   Solution: Let $D(2,3)$ be the reflection of A with respect to $y=x$, and $E(3,-2)$ the refection of A with respect to the x-axis. Then if B is on $y=x$ and C is on the x-axis, then $|AB|=|DB|$ and $|AC|=|CE|$. Then the perimeter of ΔABC is

$$|AB|+|BC|+|AC|=|DB|+|BC|+|CE|\ge|DE|=\sqrt{26}.$$

   Clearly, this lower bound can be attained.

2. Mr. Curry is playing basketball. Suppose that, on the average, he makes two successful three-point shots out of every five attempts. What is the probability that he will be able to make at least two successful 3-point shots in a row out of four attempts?

Answer: $\dfrac{44}{125}$

Solution: There are only a few possibilities to the contrary. Note that if Mr. Curry shoots at least three, for certain, two of them will be consecutive. Hence, we need consider only the cases wherein he makes two shots or less.

No shots made: The probability of this happening is simply $\left(\frac{3}{5}\right)^4 = \frac{81}{625}$.

One shot made: This one shot can be any one of the four shots, and so the probability in this case is $4 \cdot \frac{2}{5} \cdot \left(\frac{3}{5}\right)^3 = \frac{216}{625}$.

Two shots made: There are $\binom{4}{2} = 6$ ways for Mr. Curry to pick two out of four shots to make, and three of those have the two shots next to each other. Hence, the probability Steph makes two shots, not consecutively, is $3 \cdot \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2 = \frac{108}{625}$.

Adding all of these gives Mr. Curry a $\frac{81+216+108}{625} = \frac{81}{125}$ chance of not making two consecutive shots. Hence, the chance he makes two consecutive shots is $1 - \frac{81}{125} = \frac{44}{125}$.

3. Given that $100^2 + 1^2 = 65^2 + 76^2 = pq$ for some primes p and q. Find $p + q$.

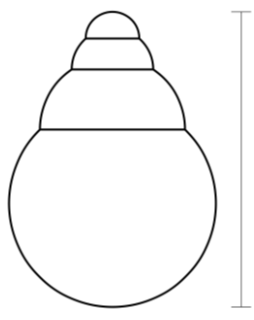
Answer: 210

Solution: Recall the identity $(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2 = (ac + bd)^2 + (ad - bc)^2$. This, and the form of the equation, plus matching parities (note that $ad+bc$ and $ad-bc$ must share the same parity) suggests, as one possible setup:

$$\begin{aligned} ad + bc &= 100 \\ ac - bd &= 1 \\ ad - bc &= 76 \\ ac + bd &= 65 \end{aligned}$$

Hence, we have $bd = 32$, $ac = 33$, $ad = 88$, $bc = 12$. Wanting integers, we note that both bc and bd are divisible by 4. Letting $b = 4$ instantly works: we have $c = 3$, $a = 11$, and $d = 8$. Plugging back into the leftmost side of the original identity, we have $(a^2 + b^2)(c^2 + d^2) = 137 \cdot 73$. Both 137 and 73, from the given, must be prime, and thus p and q are those two integers in some order.

4. Refer to the figure. A hemisphere is placed on top of a sphere having radius 2017. A second hemisphere is then placed on top of the first hemisphere, and a third hemisphere is also placed on top of the second hemisphere. All the centers are collinear, and the three hemispheres all have empty interiors and negligible width. What is the maximum height of this tower made up of the sphere and three hemispheres?



Answer: 6051

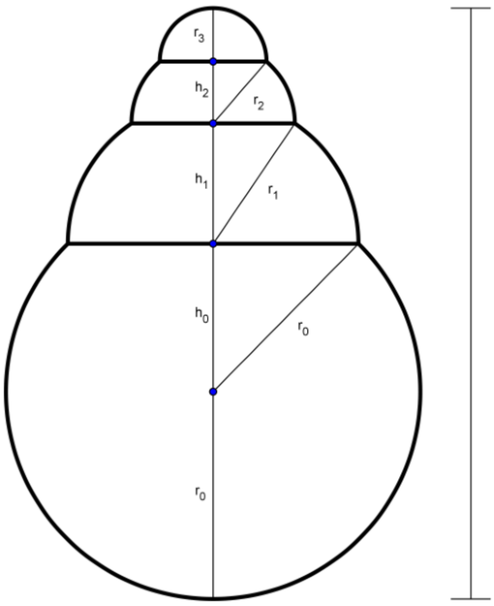
Solution: Let $r_0 = 2017$. In the figure below, $h_i^2 = r_i^2 - r_{i+1}^2$. By the Cauchy-Schwarz Inequality,

$$(h_0 + h_1 + h_2 + r_3)^2 \le 4(h_0^2 + h_1^2 + h_2^2 + r_3^2) = 4(r_0^2 - r_1^2 + r_1^2 - r_2^2 + r_2^2 - r_3^2 + r_3^2) = 4r_0^2$$

and so

$$h = r_0 + (h_0 + h_1 + h_2 + r_3) \le r_0 + 2r_0 = 3r_0 = 6051.$$

The maximum height 6051 is attained if $h_0 = h_1 = h_2 = r_3$, or equivalently, $r_0^2 - r_1^2 = r_1^2 - r_2^2 = r_2^2 - r_3^2 = r_3^2$. This can happen by setting $h_0 = h_1 = h_2 = r_3 = \frac{r_0}{2}$.



5. Let x, y, z be positive integers such that

$$\begin{aligned}(x+y)(y+z) &= 2016 \\ (x+y)(z+x) &= 1080\end{aligned}$$

Determine the smallest possible value for $x+y+z$.

Answer: 61

Solution: Note that $2016 = 2^5 \times 3^2 \times 7$ and $1080 = 2^3 \times 3^3 \times 5$. Moreover

$$x+y+z = \frac{1}{2}((x+y)+(y+z)+(z+x))$$

Since $x+y$ is a common factor for both 2016 and 1080, and we want $x+y+z$ to be as small as possible, then we try to find the largest possible factor for 2016 and 1080 such that x, y, z are integers

$$\begin{aligned}(x+y)(y+z) &= \left(2^3 \times 3^2\right) \times \left(2^2 \times 7\right) = 72 \times 28 \\ (x+y)(z+x) &= \left(2^3 \times 3^2\right) \times (3 \times 5) = 72 \times 15\end{aligned}$$

hence $x+y=72,\ y+z=28,\ z+x=15$. But $x+y+z=\frac{1}{2}(72+28+15)=57.5$ which cannot be since x, y, z are integers.

Therefore we try the following:

$$\begin{aligned}(x+y)(y+z) &= \left(2^2 \times 3^2\right) \times \left(2^3 \times 7\right) = 36 \times 56 \\ (x+y)(z+x) &= \left(2^2 \times 3^2\right) \times (2 \times 3 \times 5) = 36 \times 30\end{aligned}$$

hence $x+y+z=\frac{1}{2}(36+56+30)=61$. This is the smallest possible sum given that x, y, z are positive integers.

SPARE

1. (Easy) Define $f(x) = x^2 - x - 4$ and $g(x) = \log_2(x - 2)$. Find the domain of $g \circ f$.

   Answer: $(-\infty, -2) \cup (3, +\infty)$.

   Solution: Note first that we need $f(x) > 2$ so that $g \circ f$ is defined. Hence, we have $x^2 - x - 4 > 2$, or $x^2 - x - 6 > 0$. This factors into $(x + 2)(x - 3) > 0$, and quick interval testing shows that the intervals $(-\infty, -2)$ and $(3, +\infty)$ are the ones that work. (A table of signs may also be used.)

2. (Easy) How many real number solutions does the equation $\frac{1}{3}x^4 + 5|x| = 7$ have?

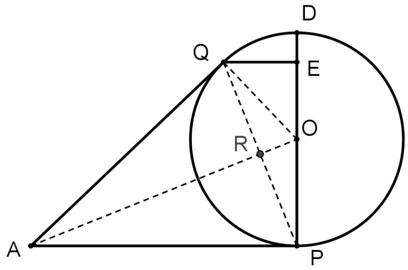
   Answer: 2

   Solution: The solutions correspond to intersection points of $y = 7 - \frac{1}{3}x^4$ amd $y = 5|x|$. Clearly, there are only two intersection points.

3. (Easy/Average) The segments AP and AQ are tangent to circle O at points P and Q, respectively. Moreover, QE is perpendicular to diameter PD of length 4. If $PE = 3.6$ and $AP = 6$, what is the length of QE?

   Answer: 1.2

   Solution: Draw AO, PQ and OQ. We note that $AO \perp PQ$ since $AQ = AP$ and $OQ = OP$.



   This implies that $\angle PAO \cong \angle EPQ$ (both complementary to $\angle APR$)

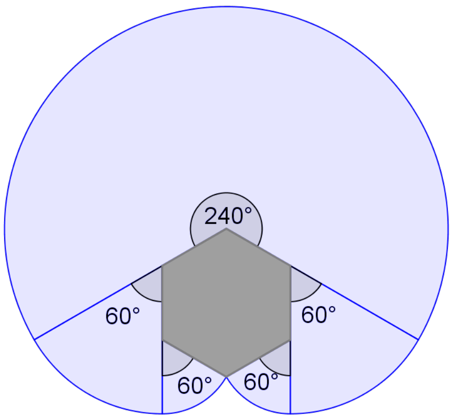
   Hence $\triangle PAO \approx \triangle EPQ$. Therefore

$$\frac{AP}{OP} = \frac{PE}{QE} \implies QE = \frac{PE \times OP}{AP} \implies QE = \frac{3.6 \times 2}{6} = 1.2$$

4. (Average) A vertical pole has a cross section of a regular hexagon with a side length of 1 foot. If a dog on a leash is tied onto one vertex, and the leash is three feet long, determine the total area that the dog can walk on.

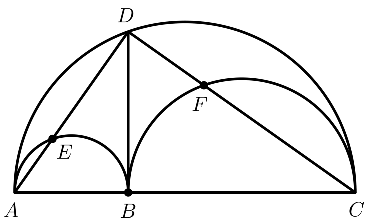
   Answer: $\frac{23}{3}\pi$ square feet

   Solution: The total area is indicated by the lightly shaded region below.



$$\begin{aligned} & \frac{240}{360}\pi(3)^2 + 2\left(\frac{60}{360}\right)\pi(2)^2 + 2\left(\frac{60}{360}\right)\pi(1)^2 \\ = & \frac{2}{3}(9\pi) + \frac{1}{3}(4\pi) + \frac{1}{3}(\pi) = \frac{23}{3}\pi \end{aligned}$$

5. (Average) The figure shows a semicircle, where B is a point on its diameter AC so that $AB = 6$ and $BC = 12$. The line perpendicular to this diameter meets the semicircle at D. The semicircles with diameters AB and BC, drawn as shown, meet AD and CD at E and F, respectively. Find the distance between E and F.



Answer: $6\sqrt{2}$

Solution: Observe that $DEFB$ is a rectangle so $EF = DB$. However, $DB^2 = AB \times BC = 72$. Thus, $EF = 6\sqrt{2}$.

6. (Average) For how many primes $p < 50$ is $p^4 + 5p^3 + 4$ divisible by 5?

Answer: 13

Solution: Clearly, the expression is not divisible by 5 when $p = 5$. For any prime p other than 5, note that $p^4 \equiv 1 \pmod 5$, so $p^4 + 5p^3 + 4 \equiv 0 \pmod 5$. Hence, the problem reduces to counting the number of primes less than 50 other than 5. There are 13 such primes.

7. (Average/Difficult) A geometric sequence has a nonzero first term, distinct terms, and a positive common ratio. If the second, fourth, and fifth terms form an arithmetic sequence, find the common ratio of the geometric sequence.

Answer: $\dfrac{1+\sqrt{5}}{2}$

Solution: Let a_1 be the first term and r the common ratio of the geometric sequence. Since the second, fourth, and fifth terms form an arithmetic sequence,

$$\begin{aligned} a_1r^3 - a_1r &= a_1r^4 - a_1r^3 \\ r^3 - r &= r^4 - r^3 \\ 0 &= r^4 - 2r^3 + r \\ 0 &= r(r^3 - 2r^2 + 1) \\ 0 &= r(r-1)(r^2 - r - 1) \\ r &= 0, 1, \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Since the geometric sequence has distinct terms and a positive common ratio, the only possible value of the common ratio is $\frac{1+\sqrt{5}}{2}$.

8. (Average/Difficult) Suppose each of five sticks is broken into a long part and a short part. The ten parts are arranged into five pairs and glued back together so that again there are five sticks. What is the probability that each long part will be paired with a short part?

Answer: $\dfrac{8}{63}$

Solution: The 5 short pieces and 5 long pieces can be lined up in a row in $\frac{10!}{5!5!}$ ways. Consider each of the 5 pairs of consecutive pieces as defining the reconstructed sticks. Each of those pairs could combine a short piece (S) and a long piece (L) in two ways: SL or LS. Therefore, the number of permutations that would produce 5 sticks, each having a short and a long component is 2^5, and so the desired probability is $2^5/\binom{10}{5} = \frac{8}{63}$.