17^{th} Philippine Mathematical Olympiad

Area Stage

15 November 2014

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. 14
2. $\sqrt[3]{4}$
3. -1
4. 2
5. $\frac{28}{3} = 9\frac{1}{3}$
6. $2 - \sqrt[3]{2}$
7. 144
8. $6!5! = 86400$
9. $\frac{\pi}{16}$
10. $70°$
11. 0, 10
12. 12
13. 6
14. 24
15. 440
16. 1007
17. 31
18. $\sqrt{60} = 2\sqrt{15}$
19. $\frac{3^5}{2^7} = \frac{243}{128}$
20. $17°$

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Arrange these four numbers from smallest to largest: $\log_3 2$, $\log_5 3$, $\log_{625} 75$, $\frac{2}{3}$.
   Solution: The numbers, arranged from smallest to largest, are $\log_3 2$, $\frac{2}{3}$, $\log_{625} 75$, and $\log_5 3$.

   - Since $\left(3^{\log_3 2}\right)^3 = 8$ and $\left(3^{\frac{2}{3}}\right)^3 = 9$, then $\log_3 2 < \frac{2}{3}$.
   - Since $\left(625^{\frac{2}{3}}\right)^3 = 5^8 = 5^6 \cdot 25$ and $\left(625^{\log_{625} 75}\right)^3 = 75^3 = 5^6 \cdot 27$, then $\frac{2}{3} < \log_{625} 75$.
   - If $A = \log_{625} 75$, then $5^{4A} = 75$. On the other hand, $5^{4\log_5 3} = 81$. Thus, $\log_{625} 75 < \log_5 3$.

2. What is the greatest common factor of all integers of the form $p^4 - 1$, where p is a prime number greater than 5?
   Solution: Let $f(p) = p^4 - 1 = (p-1)(p+1)(p^2+1)$. Note that $f(7) = 2^5 \cdot 3 \cdot 5^2$ and $f(11) = 2^4 \cdot 3 \cdot 5 \cdot 61$. We now show that their greatest common factor, $2^4 \cdot 3 \cdot 5$, is actually the greatest common factor of all numbers $p^4 - 1$ so described.

   - Since p is odd, then $p^2 + 1$ is even. Both $p - 1$ and $p + 1$ are even, and since they are consecutive even integers, one is actually divisible by 4. Thus, $f(p)$ is always divisible by 2^4.
   - When divided by 3, p has remainder either 1 or 2.
     - If $p \equiv 1$, then $3|p-1$.
     - If $p \equiv 2$, then $3|p+1$.

     Thus, $f(p)$ is always divisible by 3.

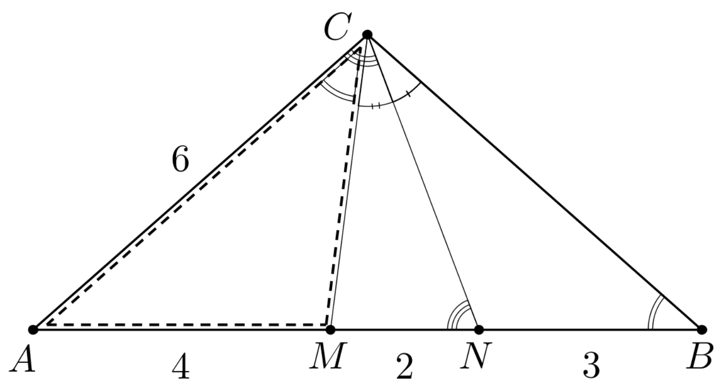
- When divided by 5, p has remainder 1, 2, 3 or 4.
  - If $p \equiv 1$, then $5|p-1$.
  - If $p \equiv 2$, then $p^2+1 \equiv 2^2+1 = 5 \equiv 0$.
  - If $p \equiv 3$, then $p^2+1 \equiv 3^2+1 = 10 \equiv 0$.
  - If $p \equiv 4$, then $5|p+1$.

  Thus, $f(p)$ is always divisible by 5.

Therefore, the greatest common factor is $2^4 \cdot 3 \cdot 5 = 240$.

3. Points A, M, N and B are collinear, in that order, and $AM = 4$, $MN = 2$, $NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.

Solution:



Since $\dfrac{CA}{AM} = \dfrac{3}{2} = \dfrac{BA}{AC}$ and $\angle CAM = \angle BAC$, then $\triangle CAM \sim \triangle BAC$. Therefore,

$$\angle MCA = \angle CBA. \tag{1}$$

Since $AC = 6 = AN$, then $\triangle CAN$ is isosceles. Therefore,

$$\angle ACN = \angle ANC. \tag{2}$$

Thus,

$$\begin{aligned}
\angle BCN &= \angle ANC - \angle CBA && \text{since } \angle ANC \text{ is an exterior angle of } \triangle BNC \\
&= \angle ACN - \angle MCA && \text{using (1) and (2)} \\
&= \angle MCN.
\end{aligned}$$