20th Philippine Mathematical Olympiad

Area Stage, 25 November 2017

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. Suppose that x and y are nonzero real numbers such that $\left(x+\frac{1}{y}\right)\left(y+\frac{1}{x}\right)=7$. Find the value of $\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{x^2}\right)$.

2. In how many ways can the integers

$$-5,-4,-3,-2,-1,1,2,3,4,5$$

be arranged in a circle such that the product of each pair of adjacent integers is negative? (Assume that arrangements which can be obtained by rotation are considered the same.)

3. Let P be a point inside the isosceles trapezoid $ABCD$ where AD is one of the bases, and let PA, PB, PC, and PD bisect angles A, B, C, and D respectively. If $PA=3$ and $\angle APD=120°$, find the area of trapezoid $ABCD$.

4. Determine the number of ordered pairs of integers (p,q) for which $p^2+q^2<10$ and $-2^p\le q\le 2^p$.

5. Let $f(x)=\sqrt{4\sin^4 x-\sin^2 x\cos^2 x+4\cos^4 x}$ for any $x\in\mathbb{R}$. Let M and m be the maximum and minimum values of f, respectively. Find the product of M and m.

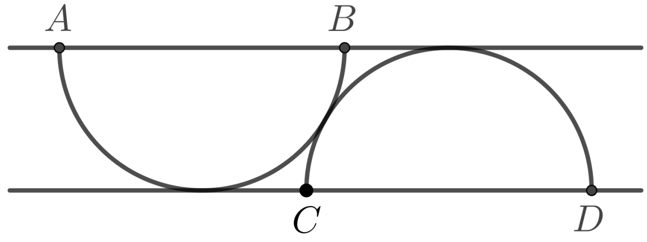
6. A semicircle Γ has diameter $AB=25$. Point P lies on AB with $AP=16$ and C is on the semicircle such that $PC\perp AB$. A circle ω is drawn so that it is tangent to segment PC, segment PB, and Γ. What is the radius of ω?

7. Determine the area of the polygon formed by the ordered pairs (x,y) where x and y are positive integers which satisfy the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{13}.$$

8. Let A be the sum of the decimal digits of the largest 2017-digit multiple of 7 and let B be the sum of the decimal digits of the smallest 2017-digit multiple of 7. Find $A-B$.

9. Two semicircles, each with radius $\sqrt{2}$, are tangent to each other, as shown in the figure below. If $AB\parallel CD$, determine the length of segment AD.



10. The boat is sinking! Passengers must then be saved, but the rescuer must know their count. If the passengers group themselves into 7, one group will only have 4 passengers. If the passengers group themselves into 11, one group will only have 7 passengers. If the passengers group themselves into 13, one group will only have 10 passengers. How many passengers are there if the boat carried at most 1000 passengers?

11. Given $a_n \in \mathbb{Z}$ with $a_{10} = 11$ and $a_9 = -143$, determine the number of polynomials of the form

$$P(x) = \sum_{n=0}^{10} a_n x^n$$

such that the zeros of $P(x)$ are all positive integers.

12. In how many ways can nine chips be selected from a bag that contains three red chips, three blue chips, three white chips, and three yellow chips? (Assume that the order of selection is irrelevant and that the chips are identical except for their color.)

13. Let L_1 be the line with equation $6x - y + 6 = 0$. Let P and Q be the points of intersection of L_1 with the x-axis and y-axis, respectively. A line L_2 that passes through the point $(1, 0)$ intersects the y-axis and L_1 at R and S, respectively. If O denotes the origin and the area of ΔOPQ is six times the area of ΔQRS, find all possible equations of the line L_2. Express your answer in the form $y = mx + b$.

14. Find the smallest positive integer whose cube ends in 2017.

15. Let $\{x_k\}_{k=1}^{n}$ be a sequence whose terms come from $\{2, 3, 6\}$. If

$$x_1 + x_2 + \cdots + x_n = 633 \quad \text{and} \quad \frac{1}{x_1^2} + \frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} = \frac{2017}{36},$$

find the value of n.

16. Let S be a subset of $\{1, 2, \ldots, 2017\}$ such that no two elements of S have a sum divisible by 37. Find the maximum number of elements that S can have.

17. In cyclic pentagon $ABCDE$, $\angle ABD = 90°, BC = CD$, and AE is parallel to BC. If $AB = 8$ and $BD = 6$, find AE^2.

18. The edges of a square are to be colored either red, blue, yellow, pink, or black. Each side of the square can only have one color, but a color may color many sides. How many different ways are there to color the square if two ways that can be obtained from each other by rotation are identical?

19. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If $a_n \lfloor a_n \rfloor = 49^n + 2n + 1$, find the value of $2S + 1$, where $S = \left\lfloor \sum_{n=1}^{2017} \frac{a_n}{2} \right\rfloor$.

20. A spider and a fly are on diametrically opposite vertices of a web in the shape of a regular hexagon. The fly is stuck and cannot move. On the other hand, the spider can walk freely along the edges of the hexagon. Each time the spider reaches a vertex, it randomly chooses between two adjacent edges with equal probability, and proceeds to walk along that edge. On average, how many edge lengths will the spider walk before getting to the fly?

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Find all pairs (r, s) of real numbers such that the zeros of the polynomials

$$f(x) = x^2 - 2rx + r$$

and

$$g(x) = 27x^3 - 27rx^2 + sx - r^6$$

are all real and nonnegative.

2. A point P is chosen randomly inside the triangle with sides 13, 20, and 21. Find the probability that the circle centered at P with radius 1 will intersect at least one of the sides of the triangle.

3. Define a sequence of integers as follows: $a_1 = 1, a_2 = 2$, and for $k \in \mathbb{N}$, $a_{k+2} = a_{k+1} + a_k$. How many different ways are there to write 2017 as a sum of distinct elements of this sequence?

Answers to the 20th PMO Area Stage

Part I. (3 points each)

1. 25
2. 2880
3. $6\sqrt{3}$
4. 17
5. $\sqrt{7}$
6. 4
7. 12096
8. 18144
9. $2(1+\sqrt{3})$
10. 634
11. 3
12. 20
13. $y=-3x+3$, $y=-10x+10$
14. 9073
15. 262
16. 991
17. 338/5
18. 165
19. $\dfrac{7(7^{2017}-1)}{6}$
20. 9

Part II. (10 points each, full solutions required)

1. Find all pairs (r,s) of real numbers such that the zeros of the polynomials

$$f(x)=x^2-2rx+r$$

and

$$g(x)=27x^3-27rx^2+sx-r^6$$

are all real and nonnegative.

Answers: $(0,0)$, $(1,9)$

Solution:

Let x_1,x_2 be the zeros of $f(x)$, and let y_1,y_2,y_3 be the zeros of $g(x)$.

By Viete's relation,

$$\begin{aligned} x_1+x_2 &= 2r \\ x_1x_2 &= r \end{aligned}$$

and

$$\begin{aligned} y_1+y_2+y_3 &= r \\ y_1y_2+y_2y_3+y_3y_1 &= \frac{s}{27} \\ y_1y_2y_3 &= \frac{r^6}{27} \end{aligned}$$

Note that

$$\left(\frac{x_1+x_2}{2}\right)^2 \geq x_1x_2 \quad \Rightarrow \quad r^2 \geq r$$

$$\begin{aligned} \frac{y_1+y_2+y_3}{3} &\geq \sqrt[3]{y_1y_2y_3} \\ \frac{r}{3} &\geq \sqrt[3]{\frac{r^6}{27}} \\ r &\geq r^2 \end{aligned}$$

Hence $r = r^2$, and consequently $x_1 = x_2$ and $y_1 = y_2 = y_3$. Moreover, $r = 0, 1$.

- If $r = 0$, then $f(x) = x^2$ with $x_1 = x_2 = 0$. And since $y_1 = y_2 = y_3$ with $y_1 + y_2 + y_3 = 0$, then ultimately $s = 0$.
- If $r = 1$, then $f(x) = x^2 - 2x + 1 = (x-1)^2$ with $x_1 = x_2 = 1$. And since $y_1 = y_2 = y_3$ with $y_1 + y_2 + y_3 = 1$ then $y_1 = y_2 = y_3 = \frac{1}{3}$. Therefore $s = 9$.

Thus, the possible ordered pairs (r, s) are $\boxed{(0,0)}$ and $\boxed{(1,9)}$.

2. A point P is chosen randomly inside the triangle with sides 13, 20, and 21. Find the probability that the circle centered at P with radius 1 will intersect at least one of the sides of the triangle.

Answer: 75/196

Solution 1: Let ABC be a triangle with sides $BC = 13, CA = 20$ and $AB = 21$ and let S be the set of points P such that the circle ω with radius 1 centered at P intersects at least one of the sides of ABC. For a fixed side of ABC (say ℓ), ω intersects ℓ if and only if P lies within one unit from ℓ. This suggests we construct a triangle $A_1B_1C_1$ such that $A_1B_1 \parallel AB, B_1C_1 \parallel BC, C_1A_1 \parallel CA$ and the corresponding parallel sides of $A_1B_1C_1$ and ABC have distance 1. Thus, the set S of such points P forms a region $\mathcal{R}$ outside $A_1B_1C_1$ but inside ABC and the probability is then the ratio of the areas of $\mathcal{R}$ and ABC.

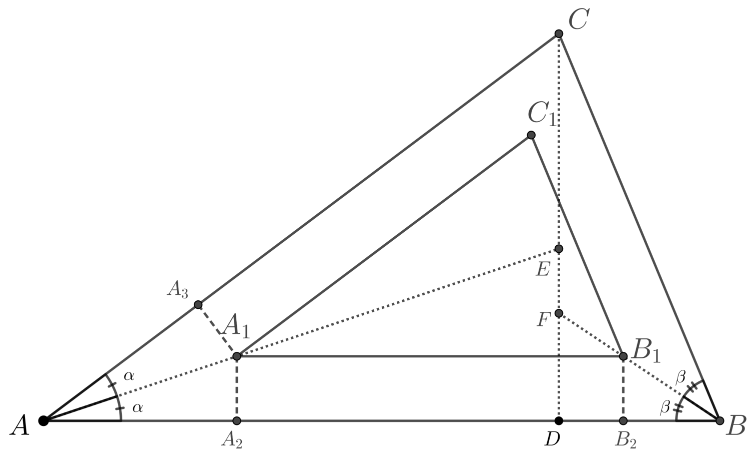
Observe that ABC and $A_1B_1C_1$ are similar and hence the ratio of their corresponding sides is constant, say $k > 0$. Also, triangle ABC is divided into four regions: the triangle $A_1B_1C_1$ and three trapezoids A_1B_1BA, B_1C_1CB and C_1A_1AC. The region $\mathcal{R}$ then comprises these trapezoids. We use $[\mathcal{P}]$ to denote the area of region $\mathcal{P}$. Using Heron's formula with semiperimeter $s = 27$, we see that $[ABC] = \sqrt{27(27-13)(27-20)(27-21)} = 126$. As ABC and $A_1B_1C_1$ are similar, $[A_1B_1C_1] : [ABC] = k^2$ and with $B_1C_1 = 13k, C_1A_1 = 20k, A_1B_1 = 21k$, we obtain

$$\begin{aligned} [ABC] &= [A_1B_1C_1] + [A_1B_1BA] + [B_1C_1CB] + [C_1A_1AC] \\ 126 &= 126k^2 + \frac{1}{2}(21k+21) + \frac{1}{2}(20k+20) + \frac{1}{2}(13k+13) \\ &= 126k^2 + 27k + 27 \end{aligned}$$

so $126k^2 + 27k - 99 = 9(k+1)(14k-11) = 0$ and $k = \frac{11}{14}$. Therefore, the probability is

$$\frac{[\mathcal{R}]}{[ABC]} = 1 - \frac{[A_1B_1C_1]}{[ABC]} = 1 - k^2 = 1 - \frac{121}{196} = \frac{75}{196}.$$

Solution 2: The additional points in the figure below (not drawn to scale) are precisely what they appear to be.



We can determine the proportionality constant k between $\triangle A_1B_1C_1$ and $\triangle ABC$ by determining $A_1B_1 = A_2B_2 = 21 - AA_2 - BB_2$. Since $\triangle AA_1A_2$ and $\triangle AA_1A_3$ are congruent right triangles, then AA_1 bisects $\angle A$. Let $\alpha = \angle A_1AA_2$. Then $\tan\alpha = \frac{A_1A_2}{AA_2} = \frac{1}{AA_2}$ so $AA_2 = \cot\alpha$.

From Heron's Formula, $[ABC] = 126$. Since $126 = \frac{1}{2}(21)(CD)$, then $CD = 12$. By the Pythagorean Theorem, $AD = 16$ and $BD = 5$.

Let the bisector of $\angle A$ meet the altitude CD at E. Thus $\frac{DE}{CE} = \frac{AD}{AC} = \frac{16}{20} = \frac{4}{5}$. Since $CE + DE = 12$, then $DE = \frac{16}{3}$. This implies $\tan\alpha = \frac{DE}{AD} = \frac{16/3}{16} = \frac{1}{3}$, and so $AA_2 = \cot\alpha = 3$.

Similarly, BB_1 bisects $\angle B$. If $\beta = \angle B_1BB_2$, then $BB_2 = \cot\beta$. Extend BB_1 to meet CD at F. Since $\frac{DF}{CF} = \frac{BD}{BC} = \frac{5}{13}$ and $CF + DF = 12$, then $DF = \frac{10}{3}$. This implies $\tan\beta = \frac{DF}{BD} = \frac{10/3}{5} = \frac{2}{3}$, and so $BB_2 = \cot\beta = 1.5$.

Finally, $A_2B_2 = 21 - 3 - 1.5 = 16.5$. Thus, $k = \frac{A_1B_1}{AB} = \frac{16.5}{21} = \frac{11}{14}$, and so

$$\frac{[ABC] - [A_1B_1C_1]}{[ABC]} = \frac{[ABC] - k^2[ABC]}{[ABC]} = 1 - k^2 = \frac{75}{196}.$$

3. Define a sequence of integers as follows: $a_1 = 1, a_2 = 2$, and for $k \in \mathbb{N}$, $a_{k+2} = a_{k+1} + a_k$. How many different ways are there to write 2017 as a sum of distinct elements of this sequence?

   Answer: 24

   Solution: Note that these a_ks are in fact the Fibonacci numbers. Denote by $f(n)$ the number of distinct ways to express a number as a sum of a_ks. Note that $2017 = 1597 + 377 + 34 + 8 + 1 = a_{15} + a_{12} + a_8 + a_5 + a_1$.

   We prove the following lemma:

$$a_1 + a_2 + \cdots + a_k = a_{k+2} - 2$$

   This follows simply from induction. It is true for $k = 1$; adding a_{k+1} to both sides and using the fact that $a_{k+1} + a_{k+2} = a_{k+3}$ gives the result.

Now, denote by $f(n)$ the number of ways to express n as a sum of distinct a_ks; we are looking for $f(2017)$. Now, note that any such sum must contain either 1597 or 987. If the sum does not contain 1597, it must certainly contain 987; otherwise, from the lemma, the sum would be at most $1 + 2 + \cdots + 610 = 1595$. Moreover, if the sum contains 987 (but not 1597), it must also contain 610; otherwise, it will be at most $987 + (1 + 2 + \cdots + 377) = 987 + 985 = 1972$.

Hence, $f(2017) = 2f(420)$.

By a similar argument, any sum of 420 must contain either 377 or 233. However, this time, it is perfectly possible for this sum to contain 233 but not 144, since $233 + (1 + 2 + 3 + \cdots + 89) = 464 > 420$. We thus have two cases to deal with.

Case 1: If the sum contains 377, then we have to compute $f(43)$. Now, note that as in the argument from earlier, any sum adding up to 43 contains either 34 or 13+21. Hence $f(43) = 2f(9)$. Repeating this argument, we get $f(9) = 2f(1) = 2$. This gives us $f(43) = 4$.

Case 2: The sum does not contain 377. In this case, the sum must contain 233. We then have to compute $f(187)$. Any sum adding up to 187 must contain either 144 or 89; moreover, if it contains 89, it must contain 55 as well. Hence, $f(187) = 2f(43) = 8$ by our previous computation.

Thus, $f(420) = 3f(43) = 12$, and $f(2017) = 24$.