21st Philippine Mathematical Olympiad

Mathematical Society of the Philippines

Department of Science and Technology - Science Education Institute

PMO Director: Ma. Nerissa Abara
Test Development Committee: Christian Paul Chan Shio, Richard Eden, Louie John Vallejo
Carlo Francisco Adajar, Gari Lincoln Chua, Russelle Guadalupe, Lu Christian Ong,
and Immanuel Gabriel Sin

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National Stage, Written Phase

26 January 2019

Time: 4.5 hours *Each item is worth 7 points.*

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(2xy) + f(f(x+y)) = xf(y) + yf(x) + f(x+y)$$

for all real numbers x and y.

<u>*Solution*</u>: The only functions are $f(x) = 0$, $f(x) = x$ and $f(x) = 2 - x$. It can be checked that these are indeed solutions.

Substituting x and $\frac{1}{2}$ for x and y respectively yields

$$f(x) + f\left(f\left(x + \frac{1}{2}\right)\right) = xf\left(\frac{1}{2}\right) + \frac{1}{2}f(x) + f\left(x + \frac{1}{2}\right)$$

On the other hand, substituting $x + \frac{1}{2}$ and 0 for x and y respectively yields

$$f(0) + f\left(f\left(x + \frac{1}{2}\right)\right) = \left(x + \frac{1}{2}\right)f(0) + f\left(x + \frac{1}{2}\right)$$

Subtracting the two equations then yields

$$f(x) - f(0) = 2xf\left(\frac{1}{2}\right) - 2xf(0) \implies f(x) = 2x\left[f\left(\frac{1}{2}\right) - f(0)\right] + f(0),$$

which implies that $f(x)$ is linear. Substituting $f(x) = ax + b$ to the given functional equation then gives the three answers. ■

2. Twelve students participated in a theater festival consisting of n different performances. Suppose there were six students in each performance, and each pair of performances had at most two students in common. Determine the largest possible value of n.

<u>*Solution*</u>: We label the students by $1, 2, \ldots, 12$ and the performances by the subsets $P_1, \ldots, P_n$ of $\{1, \ldots, 12\}$. Then the problem now reduces to finding the maximum value of n such that

(a) $|P_i| = 6$ for all $1 \le i \le n$, and

(b) $|P_i \cap P_j| \le 2$ for all $1 \le i < j \le n$.

We make a $12 \times n$ $\{0,1\}$-matrix M whose entries are defined as follows:

$$M_{ij} = \begin{cases} 1 & \text{if student } i \text{ plays in performance } P_j, \\ 0 & \text{if student } i \text{ does not play in performance } P_j. \end{cases}$$

For each $i \in \{1, \dots, 12\}$, let $r_i = \sum_{j=1}^{n} M_{ij}$ be the number of times i appears in the sets $P_1, \dots, P_n$. Then, by double-counting, we have $\sum_{i=1}^{12} r_i = 6n$. Let $\mathcal{R}$ be the set of all unordered pairs of 1's that lie in the same row. Counting by rows, we see that in the ith row, there are r_i 1's and thus $\binom{r_i}{2}$ pairs. Thus, $|\mathcal{R}| = \sum_{i=1}^{12} \binom{r_i}{2}$. Counting by columns, we note that for any two columns, there are at most 2 pairs of 1's among these columns, so $|\mathcal{R}| \le 2\binom{n}{2} = n(n-1)$. Thus,

$$\sum_{i=1}^{12} \binom{r_i}{2} \le n(n-1) \implies \sum_{i=1}^{12} r_i^2 - \sum_{i=1}^{12} r_i \le 2n(n-1) \implies \sum_{i=1}^{12} r_i^2 \le 2n^2 + 4n.$$

By the Cauchy-Schwarz inequality,

$$36n^2 = \left(\sum_{i=1}^{12} r_i \right)^2 \le 12 \sum_{i=1}^{12} r_i^2 = 24n^2 + 48n,$$

which implies that $n \le 4$. For $n = 4$, we have the following specific sets $P_1, \dots, P_4$ satisfying the conditions of the problem:

$$P_1 = \{1,2,3,4,5,6\}, \quad P_2 = \{1,3,7,8,11,12\}$$
$$P_3 = \{2,4,7,8,9,10\}, \quad P_4 = \{5,6,9,10,11,12\}.$$

Hence, the maximum value of n is $n = 4$. ■

3. Find all triples (a, b, c) of positive integers such that

$$a^2 + b^2 = n \operatorname{lcm}(a,b) + n^2$$
$$b^2 + c^2 = n \operatorname{lcm}(b,c) + n^2$$
$$c^2 + a^2 = n \operatorname{lcm}(c,a) + n^2$$

for some positive integer n.

Solution: We claim that the only triples that satisfy the system are those of the form (k, k, k). It can be easily checked that all such triples are solutions, where $n = k$. Conversely, suppose that (a, b, c) is a solution. We then need to show that $a = b = c$.

Suppose that there exists some integer $d > 1$ such that $d|a, d|b, d|c$. From any of the equations of the system, we also get $d|n$. Thus, by replacing (a, b, c) with $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$, we obtain a new solution, where n is replaced by $\frac{n}{d}$. Thus, WLOG, we can assume that a, b, c, and n share no common divisor other than 1. By solving the system of equations for $2a^2$, we get

$$2a^2 = n(\operatorname{lcm}(a, b) - \operatorname{lcm}(b, c) + \operatorname{lcm}(c, a) + n).$$

Hence $n|2a^2$, and similarly, $n|2b^2$, and $n|2c^2$. But as a, b, c and n share no common divisor other than 1, it then follows that either $n = 1$ or $n = 2$.

If $n = 1$, then we have $a^2+b^2 = \operatorname{lcm}(a, b)+1$, which implies that $2ab \leq ab+1$. This gives $a = b = c = 1$, which leads to the family of solutions (k, k, k).

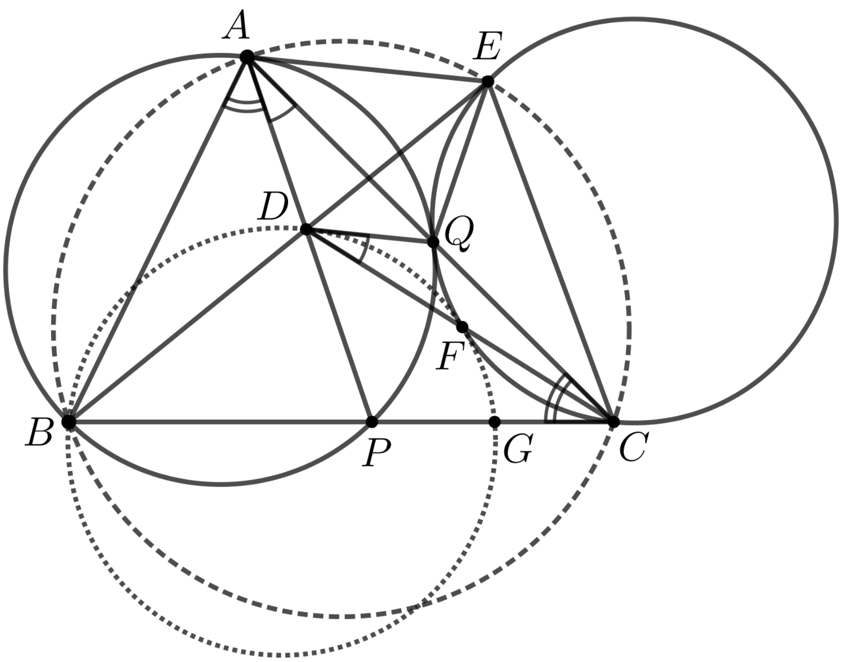
If $n = 2$, then $a^2 + b^2 = 2\operatorname{lcm}(a, b) + 4 \leq 2ab + 4$ so $(a - b)^2 \leq 4$, and $|a - b| \leq 2$. Similarly, $|b - c| \leq 2$ and $|c - a| \leq 2$. Note that no two of a, b, and c can be consecutive. To see this, suppose WLOG that $a = b + 1$. Substituting this to the first equation gives $1 = 4$. Contradiction.

Thus, at least two of a, b, and c must be equal. Without loss of generality, assume that $a = b$. Substituting to the first equation, we obtain $a = b = 2$. Thus, $4 + c^2 = 2\operatorname{lcm}(2, c) + 4$, and so c is even. This is a contradiction since we assumed that a, b, c, and n have no common divisor other than 1. Therefore, the case $n = 2$ does not give any additional solution. ■

4. In acute triangle ABC with $\angle BAC > \angle BCA$, let P be the point on side BC such that $\angle PAB = \angle BCA$. The circumcircle of triangle APB meets side AC again at Q. Point D lies on segment AP such that $\angle QDC = \angle CAP$. Point E lies on line BD such that $CE = CD$. The circumcircle of triangle CQE meets segment CD again at F, and line QF meets side BC at G. Show that B, D, F, and G are concyclic.

Solution: Refer to the figure shown below. Since $ABPQ$ is cyclic, we have $CP \cdot CB = CQ \cdot AC$. Also, we have $\triangle CAD \sim \triangle CDQ$, so $CD^2 = CQ \cdot AC$. This means that $CE^2 = CD^2 = CQ \cdot AC = CP \cdot CB$, so $\triangle CDP \sim \triangle CBD$ and $\triangle CEQ \sim \triangle CAE$. Thus, $\angle CBD = \angle CDP$ and, since $QECF$ is cyclic, $\angle CAE = \angle CEQ = \angle QFD$. Now, we see that

$$\begin{aligned}\angle EDC &= \angle CBD + \angle DCB = \angle CBD + \angle ACB - \angle ACD \\ &= \angle CBD + \angle ACB - (\angle CDP - \angle DAC) \\ &= \angle BAP + \angle DAC = \angle BAC.\end{aligned}$$



Since triangle DCE is isosceles with $CD = CE$, we get $\angle DEC = \angle BAC$. It follows that $BAEC$ is cyclic, so $\angle GBD = \angle CBD = \angle CAE$. But $\angle CAE = \angle QFD$, so $\angle GBD = \angle QFD$ and therefore, $BDFG$ is cyclic. The desired conclusion follows. ■

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Richard Eden   Christian Paul Chan Shio

Carlo Francisco Adajar
Gari Lincoln Chua
Jerome Dimabayao
Kyle Patrick Dulay
Russelle Guadalupe
Lu Christian Ong
Albert John Patupat
Diane Christine Pelejo
Carl Joshua Quines
Hazel Joy Shi
Timothy Robin Teng
Mikaela Angelina Uy
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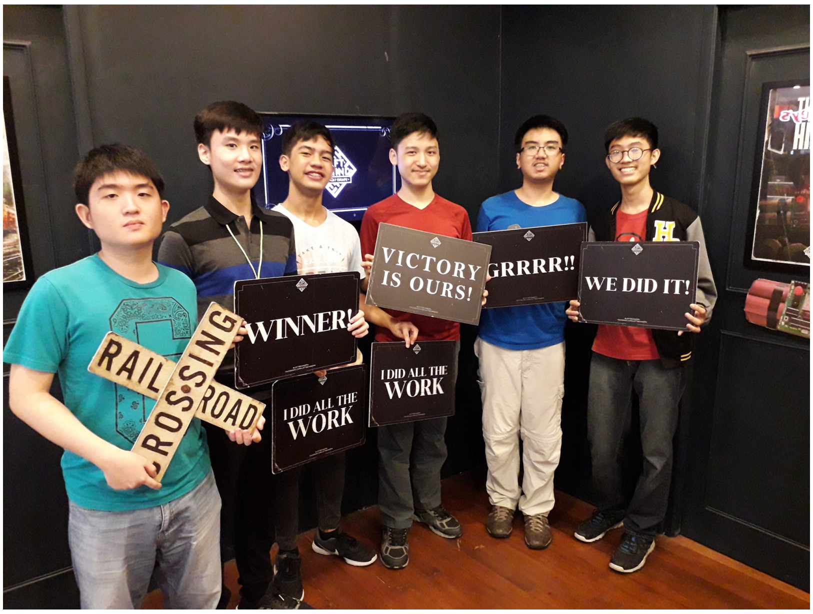

Left to Right

1. Sean Anderson Ty, *Zamboanga Chong Hua High School*
2. Dion Stephan Ong, *Ateneo de Manila Senior High School*
3. Andres Rico Gonzales III, *De La Salle University Integrated School*
4. Bryce Ainsley Sanchez, *Grace Christian College*
5. Vincent Dela Cruz, *Valenzuela City School of Mathematics and Science*
6. Immanuel Josiah Balete, *St. Stephen's High School*

Team Leader: Richard Eden, *Ateneo de Manila University*

Deputy Team Leader: Christian Paul Chan Shio, *Ateneo de Manila University*

Observer B: Russelle Guadalupe, *University of the Philippines - Diliman*