19

PHILIPPINE MATHEMATICAL OLYMPIAD

NATIONAL STAGE

21 January 2017
NISMED Auditorium
University of the Philippines
Diliman, Quezon City

SCHEDULE

7:00 AM - 7:30 AM	Registration
7:30 AM - 12:00 NN	Written Phase
12:00 NN - 1:00 PM	Lunch Break
1:00 PM - 5:00 PM	Oral Phase
6:00 PM - 8:30 PM	Dinner and Awarding Ceremonies

ABOUT THE PMO

First held in **1984**, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: **(1)** to awaken greater interest in and promote the appreciation of mathematics among students and teachers; **(2)** to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; **(3)** to provide a vehicle for the professional growth of teachers; and **(4)** to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the **International Mathematical Olympiad (IMO).** It is followed by the **Mathematical Olympiad Summer Camp (MOSC)**, a five-phase program for the twenty two national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the nineteenth since 1984. Four thousand five hundred thirty three (4533) junior and senior high school students from all over the country took the qualifying examination, out of these, two hundred twelve (212) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty and these twenty students will compete for the top three positions and hopefully move on to represent the country in the **58th IMO**, which will be held in **Rio de Janeiro, Brazil**, from **12 to 23 of July 2017** .

MESSAGE

A blessed new year!

It is always fitting to welcome the year with the highly anticipated final leg of the Philippine Mathematical Olympiad (PMO)—without a doubt, the most prestigious mathematics competition in the country.

For 19 years, the PMO has constantly produced the most outstanding talents in mathematics in the country and even in the world. These students not only represent the best in their classes, former participants have even become leaders in their respective fields whether in the academe, the government, and the industry. The PMO has also been the training grounds for would-be medalists in the so-called World Cup of mathematics, the International Mathematical Olympiad (IMO). One can simply look back at the past five years and say that the Philippines is definitely on the rise. The sight of winning our very first gold medals in last year's IMO remains fresh and we surely want to build on this victory. The PMO presents the best preparation for our top bets both in the said competition and in their respective career paths.

As we always say, the high degree of competition in the PMO brings out the paramount potential of our students in mathematics. To this, the Science Education Institute will continue to support the Mathematics Society of the Philippines in nurturing the Filipino youth who have unquestionable potential to become our future leaders. We are positive that this year's PMO will again produce a new breed of difference makers in the science and technology landscape.

We look forward to an exciting PMO and we wish all the contestants the best.

JOSETTE T. BIYO, PH. D.

Director

Science Education Institute, DOST

MESSAGE

My warmest greetings and congratulations to the twenty-two participants in the National Finals of the 19th Philippine Mathematical Olympiad. You have bested 4,533 participants from 389 schools. But after this stage, another journey commences as you train for possible participation in the International Mathematical Olympiad.

In the past years, our aim to "*Go for Gold*" has inspired us to work harder and perform better in the IMO. In 2016, the Philippines finally got the much-desired gold (not one, but two), achieving our highest team score and country rank since we started participating in the IMO. This brings much pressure to this year's IMO team. I am confident that we have the best trainors to guide and hone your talents further. More importantly, you should always remember that the PMO and IMO are not only venues to attain awards and honors, but also are venues for you to do your best, enjoy math, and promote camaraderie among math enthusiasts.

I hope that your experience in math competitions will bear a strong positive imprint to guide you in your future aspirations. I also hope that your interest and exceptional talent in mathematics will inspire you to become excellent mathematicians, scientists, or engineers who will help our country advance in science and technology.

My most heartfelt gratitude to the dedicated coaches, supportive school administrators, generous sponsors, and of course, the loving parents. Your contribution to the success of our young math enthusiasts is priceless.

The Mathematical Society of the Philippines is committed to promoting and enhancing mathematics in our country and it is an honor to once again conduct the 19th PMO in partnership with the Department of Science and Technology.

Mabuhay tayong lahat!

MARIAN P. ROQUE, PH. D.

President

Mathematical Society of the Philippines

MESSAGE

To God be the Glory!

With much dedication, the Philippine Math Olympiad quest has finally geared the Filipino youth towards global excellence by winning our first two gold medals in the recent 2016 International Math Olympiad. Congratulations!

May this New Year 2017 bring more aspirations to leap further, "*the greater the sacrifice, the greater the reward.*"

My deep regards to all the parents who continually give full support for the success of their children.

Success is about each one doing his or her part.

Romans 8:28

"And we know that all things work together for good to those who love God, to those who are called according to His purpose."

LUCERO ONG

Assistant Vice President

Sharp Calculators

Collins International Trading Corp.

MESSAGE

On behalf of the Foundation for Upgrading the Standard of Education (FUSE), I would like to congratulate the top 3 finishers of the 19th Philippine Mathematical Olympiad (PMO), the country's oldest and most prestigious math competition among students in the public and private high schools.

Hats off, too, to the over 200 qualifiers who advanced to the area stage and the more than 4,000 students nationwide who participated and tried their luck in the qualifying phase of this annual contest organized by the Mathematical Society of the Philippines (MSP).

I hope and pray that the country will finally win the gold medal in the next International Mathematical Olympiad, a feat that has eluded us since we joined the event in the late 80s. But we are getting close as shown by the remarkable showing--36th place among 104 countries—of the Philippines' representatives to the Changmai, Thailand Olympiad in 2015.

While we pursue our highest goals, the fact remains the men and women behind PMO have done wonders in promoting and developing mathematics education, and arousing the interests of students and teachers and getting them to appreciate that math is not a difficult and intimidating subject, after all.

FUSE shares the same vision and as an active player, alongside PMO, MSP and other stakeholders, nothing is beyond reach in our separate, but relevant initiatives to raise the level of mathematics learning in our country.

I wish more success to the winners as they journey to the next Olympiad, and my deepest gratitude to the organizers for transforming the event into a vehicle for growth for high school students and math teachers.

Happy New Year.

DR. LUCIO C. TAN

Vice-Chairman, Board of Trustees

Foundation for Upgrading the Standard of Education Inc.

MESSAGE

THE PMO TEAM

DIRECTOR

Ma. Nerissa Masangkay Abara

ASSISTANT DIRECTORS

Jasmin-Mae Santos
Rigor Ponsones
Recto Rex Calingasan

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Johnatan Pimentel
Victoria May Paguio

SECRETARY

May Anne Tirado
Gaudella Ruiz

LAYOUT DESIGNER

Rolando Perez III

TEST DEVELOPMENT COMMITTEE

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Carlo Francisco Adajar
John Gabriel Pelias
Timothy Robin Teng
Mark Anthony Tolentino
Jade Ventura
Louie John Vallejo
Richard Eden

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Dennis Leyson
Renier Mendoza
Diane Christine Pelejo
Arrianne Crystal Velasco

REGIONAL COORDINATORS

REGION I, CAR
Jerico Bacani

REGION II
Crizaldy Binarao

REGION III
Marites Batac

REGION IV-A
Sharon Lubag

REGION IV-B
Shiela Grace Soriano

REGION V
Joel Noche

REGION VI
Ferdinand Monoso

REGION VII
Lorna Almocera

REGION VIII
Ariel Salarda

REGION IX
Rochelleo Mariano

REGION X, XII, ARMM
Gina Malacas

REGION XI
Eveyth Deligero

REGION XIII
Miraluna Herrera

NCR
Len Patrick Dominic Garces

AREA STAGE WINNERS

LUZON

1	**Albert John Patupat**	De La Salle University Integrated School
2	**Vince Jan Torres**	Santa Rosa Science and Technology High School
3	**Emmanuel Osbert Cajayon**	Emilio Aguinaldo College

VISAYAS

1	**Cris Jericho Cruz**	Philippine Science High School - W. Visayas Campus
2	**Makarios Joash Wee**	Philippine Christian Gospel School
3	**Jonathan Conrad Yu**	Philippine Christian Gospel School

MINDANAO

1	**Sean Anderson Ty**	Zamboanga Chong Hua High School
2	**Xavier Jefferson Ray Go**	Zamboanga Chong Hua High School
3	**Fedrick Lance Lim**	Zamboanga Chong Hua High School

NCR

1	**Kyle Patrick Dulay**	Philippine Science High School - Main Campus
2	**Farrell Eldrian Wu**	MGC New Life Christian Academy
3	**Clyde Wesley Ang**	Chiang Kai Shek College

PRIZES

The prizes for the TOP THREE for each Area/Region (Luzon, Visayas, Mindanao, NCR) are:

FIRST PLACE - Medal and SHARP Calculator

SECOND PLACE - Medal and SHARP Calculator

THIRD PLACE - Medal and SHARP Calculator

The prizes for the TOP THREE in the NATIONAL FINAL STAGE are:

CHAMPION - P 20,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

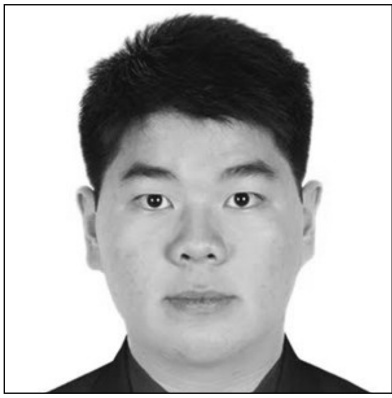
FIRST RUNNER-UP - P 15,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

SECOND RUNNER-UP - P 10,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

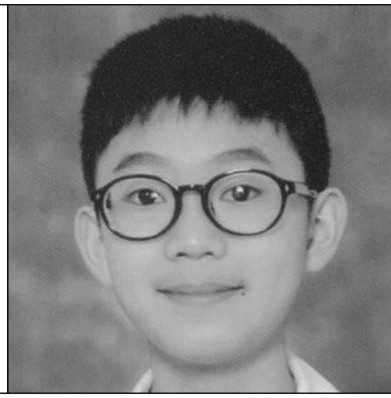
The coaches of the top three will receive P 5,000, P 3,000, and P 2,000, respectively.

The schools of the top three will receive trophies.

NATIONAL FINALISTS

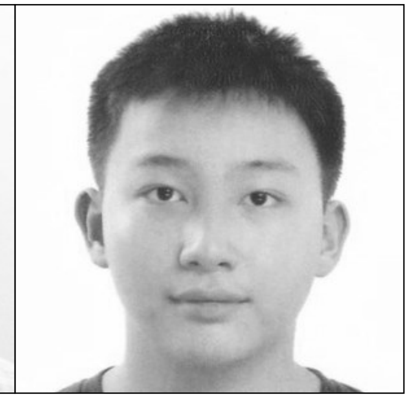


CLYDE WESLEY
ANG
Chiang Kai Shek College



IMMANUEL JOSIAH
BALETE
St. Stephen's High School

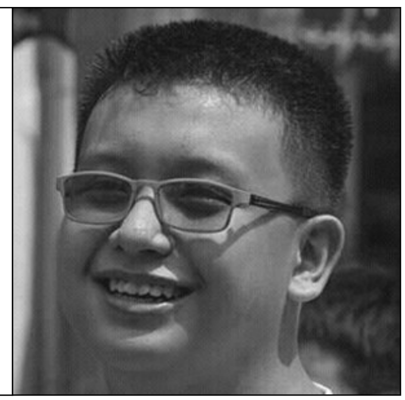
KIRK PATRICK
BAMBA
Mataas na Paaralang
Neptali A. Gonzales



LUKE MATTHEWS
BERNARDO
De La Salle University
Integrated School - Manila

EMMANUEL OSBERT
CAJAYON
Emilio Aguinaldo College

JINGER
CHONG
Saint Jude Catholic School

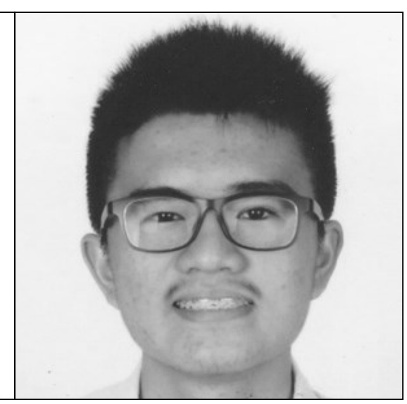


ELIJAMIN WOLFGANG
CLAVERIA
Philippine Science High
School - Main Campus

VINCENT
DELA CRUZ
Valenzuela City School of
Mathematics and Science

KYLE PATRICK
DULAY
Philippine Science High
School - Main Campus

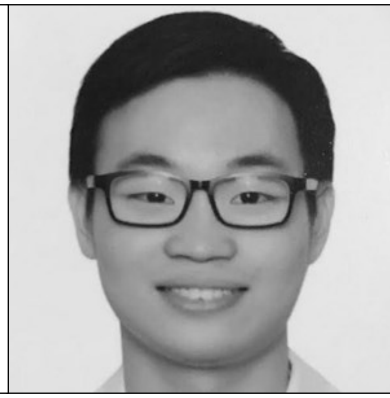
CHRISTIAN PHILIP
GELERA
Philippine Science High
School – Main Campus



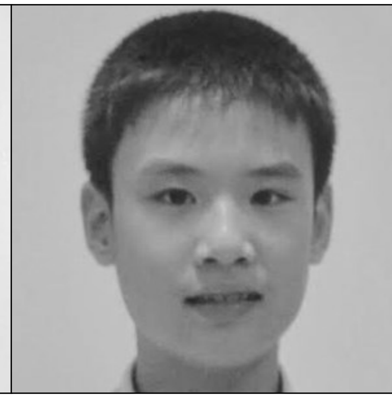
XAVIER JEFFERSON RAY
GO
Zamboanga Chong Hua
High School

NATIONAL FINALISTS

ANDRES RICO
GONZALES III
Colegio de San Juan de Letran



SEDRICK SCOTT
KEH
Xavier School



DION STEPHAN
ONG
Ateneo de Manila
Junior High School

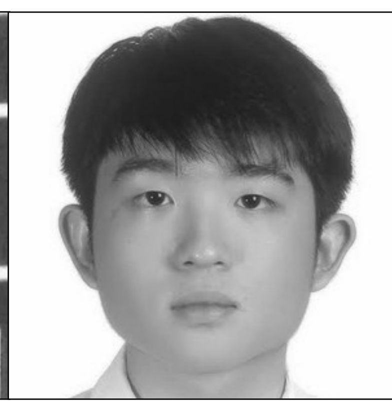
STEFAN MARCUS
ONG
Saint Jude Catholic School

ALBERT JOHN
PATUPAT
De La Salle University

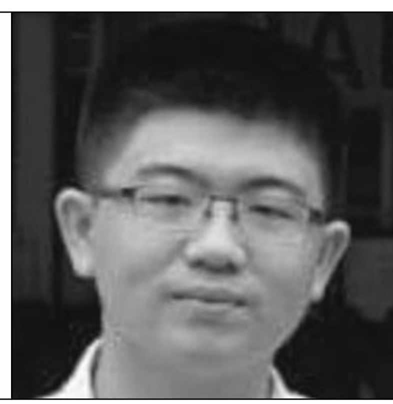
CARL JOSHUA
QUINES
Valenzuela City School of
Mathematics and Science

RAFAEL JOSE
SANTIAGO
Philippine Science High
School - Main Campus

VINCE JAN
TORRES
Santa Rosa Science and
Technology High School



SEAN ANDERSON
TY
Zamboanga Chong Hua
High School



STEVEN JOHN
WANG
Uno High School

FARRELL ELDRIAN
WU
MGC New Life
Christian Academy

18TH PMO HIGHLIGHTS

19th Philippine Mathematical Olympiad

Qualifying Stage

22 October 2016

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $27^3 + 27^3 + 27^3 = 27^x$, what is the value of x?

   (a) $\frac{10}{3}$ (b) 4 (c) 9 (d) 12

2. Let $a, b > 0$. If $|x - a| \leq a + b$, then what is the minimum value of x?

   (a) $-2a - b$ (b) $-a - b$ (c) a (d) $-b$

3. One diagonal of a rhombus is three times as long as the other. If the rhombus has an area of 54 square meters, what is its perimeter?

   (a) 9 meters (b) $12\sqrt{10}$ meters (c) 36 meters (d) $9\sqrt{5}$ meters

4. Suppose that r_1 and r_2 are the roots of the equation $4x^2 - 3x - 7 = 0$. What is the sum of the squares of the reciprocals of r_1 and r_2?

   (a) $-\frac{3}{7}$ (b) $-\frac{47}{49}$ (c) $\frac{65}{49}$ (d) $\frac{6}{7}$

5. The lengths of the sides of a triangle are 3, 5, and x. The lengths of the sides of another triangle are 4, 6, and y. If the lengths of all sides of both triangles are integers, what is the maximum value of $|x - y|$?

   (a) 2 (b) 6 (c) 7 (d) 8

6. Three circles with radii 4, 5, and 9 have the same center. If $x\%$ of the area of the largest circle lies between the other two circles, what is x to the nearest integer?

   (a) 9 (b) 11 (c) 25 (d) 33

7. Issa has an urn containing only red and blue marbles. She selects a number of marbles from the urn at random and without replacement. She needs to draw at least N marbles in order to be sure that she has at least two red marbles. In contrast, she needs three times as much in order to be sure that she has at least two blue marbles. How many marbles are there in the urn?

   (a) $4N - 4$ (b) $4N - 3$ (c) $4N - 2$ (d) $4N$

8. How many positive divisors of 30^9 are divisible by 400,000?

(a) 72 (b) 150 (c) 240 (d) 520

9. Evaluate the following sum:

$$1 + \cos\frac{\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{3\pi}{3} + \cos\frac{4\pi}{3} + \ldots + \cos\frac{2016\pi}{3}$$

(a) 1 (b) 0 (c) -1 (d) $\frac{1}{2}$

10. An infinite geometric series has first term 7 and sum between 8 and 9, inclusive. Find the sum of the smallest and largest possible values of its common ratio.

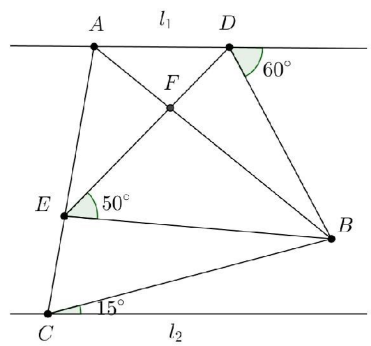
(a) 15/56 (b) 23/56 (c) 17/72 (d) 25/72

11. When $2a$ is divided by 7, the remainder is 5. When $3b$ is divided by 7, the remainder is also 5. What is the remainder when $a + b$ is divided by 7?

(a) 2 (b) 3 (c) 5 (d) 6

12. In the figure on the right (not drawn to scale), triangle ABC is equilateral, triangle DBE is isosceles with $ED = BD$, and the lines l_1 and l_2 are parallel. What is $m\angle FBE$?

(a) 30° (b) 35° (c) 40° (d) 45°



13. How many three-digit numbers have distinct digits that add up to 21?

(a) 18 (b) 24 (c) 30 (d) 36

14. A regular hexagon with area 28 is inscribed in a circle. What would the area of a square inscribed in the same circle be?

(a) $28\sqrt{3}$ (b) $\frac{56}{\sqrt{6}}$ (c) $\frac{112}{3\sqrt{3}}$ (d) $28\sqrt{6}$

15. A positive integer n is a *triangular number* if there exists some positive integer k for which it is the sum of the first k positive integers, that is, $n = 1 + 2 + \cdots + (k-1) + k$. How many triangular numbers are there which are less than 2016?

(a) 61 (b) 62 (c) 63 (d) 64

PART II. Choose the best answer. Each correct answer is worth three points.

1. I have 2016 identical marbles. I plan to distribute them equally into one or more identical containers. How many ways can this be done if I have an unlimited number of containers?

   (a) 10 (b) 36 (c) 1008 (d) 6552

2. Suppose that the seven-digit number $159aa72$ is a multiple of 2016. What is the sum of its distinct prime divisors?

   (a) 12 (b) 17 (c) 23 (d) 36

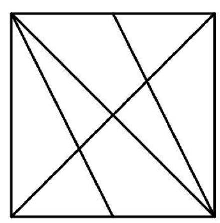
3. Let $f(x)$ be a polynomial of degree 4 with integer coefficients, leading coefficient 1, and having $\sqrt{10}+\sqrt{11}$ as one of its zeros. What is $f(1)$?

   (a) -44 (b) -40 (c) -36 (d) -21

4. If $b > 1$, find the minimum value of $\dfrac{9b^2 - 18b + 13}{b-1}$.

   (a) 0 (b) 9 (c) 12 (d) 36

5. How many triangles are there in the figure below?



   (a) 14 (b) 16 (c) 18 (d) 20

6. What is the 100th digit of the following sequence?

   $$1\ 4\ 9\ 16\ 25\ 36\ 49\ 64\ 81\ 100\ \ldots$$

   (a) 6 (b) 7 (c) 8 (d) 9

7. Louie plays a game where he throws a circular coin with radius 1 unit, which falls flat entirely inside a square board having side 10 units. He wins the game if the coin touches the boundary or the interior of a circle of radius 2 units drawn at the center of the board. What is the probability that Louie wins the game?

   (a) $\dfrac{9\pi}{64}$ (b) $\dfrac{16\pi}{81}$ (c) $\dfrac{\pi}{9}$ (d) $\dfrac{9\pi}{100}$

8. Guido and David each randomly choose an integer from 1 to 100. What is the probability that neither integer is the square of the other?

(a) 0.81 (b) 0.99 (c) 0.9919 (d) 0.9981

9. How many ordered triples of positive integers (x, y, z) are there such that $x + y + z = 20$ and two of x, y, z are odd?

(a) 135 (b) 138 (c) 141 (d) 145

10. Suppose that $x < 0 < y < 1 < z$. Which of the following statements is true?

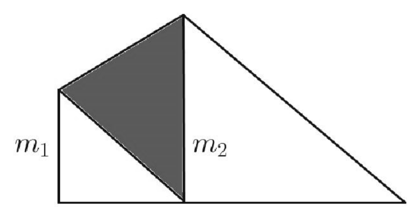
I. $\dfrac{xz - y}{x}$ is always greater than $x + yz$

II. $xy + z$ is always greater than $\dfrac{z - xy}{x}$

(a) I only (b) II only (c) both I and II (d) neither I nor II

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. A paper cut-out in the shape of an isosceles right triangle is folded in such a way that one vertex meets the edge of the opposite side, and that the constructed edges m_1 and m_2 are parallel to each other (refer to figure below, which is not drawn to scale). If the length of the triangle's leg is 2 units, what is the area of the shaded region?



2. Using the numbers 1, 2, 3, 4, 5, 6, and 7, we can form $7! = 5040$ 7-digit numbers in which the 7 digits are all distinct. If these numbers are listed in increasing order, find the 2016th number in the list.

3. Let G be the set of ordered pairs (x, y) such that (x, y) is the midpoint of $(-3, 2)$ and some point on the circle $(x + 3)^2 + (y - 1)^2 = 4$. What is the largest possible distance between any two points in G?

4. Let $f(x) = \sqrt{-x^2 + 20x + 400} + \sqrt{x^2 - 20x}$. How many elements in the range of f are integers?

5. For every positive integer n, let $s(n)$ denote the number of terminal zeroes in the decimal representation of $n!$. For example, $10! = 3,628,800$ ends in two zeroes, so $s(10) = 2$. How many positive integers less than or equal to 2016 cannot be expressed in the form $n + s(n)$ for some positive integer n?

19th Philippine Mathematical Olympiad

Qualifying Stage

22 October 2016

Part I. (2 points each)

1. A
2. D
3. B
4. C
5. B
6. B
7. A
8. B
9. A
10. D
11. B
12. B
13. A
14. C
15. B

Part II. (3 points each)

1. B
2. C
3. B
4. C
5. D
6. D
7. A
8. D
9. A
10. A

Part III. (6 points each)

1. $(6\sqrt{2} - 8)$ sq. units
2. $3,657,421$
3. 2
4. 9
5. 401

19th Philippine Mathematical Olympiad

Area Stage

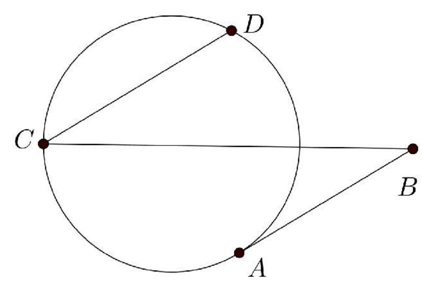
19 November 2016

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. The vertices of a triangle are at the points $(0,0)$, (a,b), and $(2016-2a,0)$, where $a>0$. If (a,b) is on the line $y=4x$, find the value(s) of a that maximizes the triangle's area.

2. Let f be a real-valued function such that
$$f(x-f(y)) = f(x) - xf(y)$$
for any real numbers x and y. If $f(0)=3$, determine $f(2016)-f(2013)$.

3. In the figure on the right, AB is tangent to the circle at point A, BC passes through the center of the circle, and CD is a chord of the circle that is parallel to AB. If $AB=6$ and $BC=12$, what is the length of CD?



4. Suppose that S_k is the sum of the first k terms of an arithmetic sequence with common difference 3. If the value of $\dfrac{S_{3n}}{S_n}$ does not depend on n, what is the 100th term of the sequence?

5. In parallelogram $ABCD$, $AB=1$, $BC=4$, and $\angle ABC = 60°$. Suppose that AC is extended from A to a point E beyond C so that triangle ADE has the same area as the parallelogram. Find the length of DE.

6. Find the exact value of $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) + \tan^{-1}\left(\dfrac{1}{8}\right)$.

7. A small class of nine boys are to change their seating arrangement by drawing their new seat numbers from a box. After the seat change, what is the probability that there is only one pair of boys who have switched seats with each other and only three boys who have unchanged seats?

8. For each $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of x in its decimal representation. For instance, $\{3.4\} = 3.4-3 = 0.4$, $\{2\}=0$, and $\{-2.7\} = -2.7-(-3) = 0.3$. Find the sum of all real numbers x for which $\{x\} = \dfrac{1}{5}x$.

9. Find the integer which is closest to the value of $\dfrac{1}{\sqrt[6]{5^6+1}-\sqrt[6]{5^6-1}}$.

10. A line intersects the y-axis, the line $y = 2x + 2$, and the x-axis at the points A, B, and C, respectively. If segment AC has a length of $4\sqrt{2}$ units and B lies in the first quadrant and is the midpoint of segment AC, find the equation of the line in slope-intercept form.

11. How many real numbers x satisfy the equation

$$\left(|x^2-12x+20|^{\log x^2}\right)^{-1+\log x} = |x^2-12x+20|^{1+\log(1/x)}?$$

12. Let $n = 2^{23}3^{17}$. How many factors of n^2 are less than n, but do not divide n?

13. A circle is inscribed in a 2 by 2 square. Four squares are placed on the corners (the spaces between circle and square), in such a way that one side of the square is tangent to the circle, and two of the vertices lie on the sides of the larger square. Find the total area of the four smaller squares.

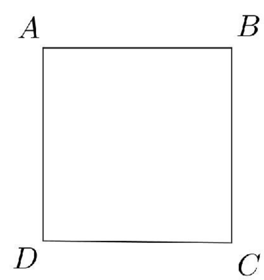
14. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (2x - y, x + 2y)$. Let $f^0(x, y) = (x, y)$ and, for each $n \in \mathbb{N}$, $f^n(x, y) = f(f^{n-1}(x, y))$. Determine the distance between $f^{2016}\left(\frac{4}{5}, \frac{3}{5}\right)$ and the origin.

15. How many numbers between 1 and 2016 are divisible by exactly one of 4, 6, or 10?

16. Let N be a natural number whose base-2016 representation is ABC. Working now in base-10, what is the remainder when $N - (A + B + C + k)$ is divided by 2015, if $k \in \{1, 2, \ldots, 2015\}$?

17. Find the number of pairs of positive integers (n, k) that satisfy the equation $(n+1)^k - 1 = n!$.

18. A railway passes through four towns A, B, C, and D. The railway forms a complete loop, as shown on the right, and trains go in both directions. Suppose that a trip between two adjacent towns costs one ticket. Using exactly eight tickets, how many distinct ways are there of traveling from town A and ending at town A? (*Note that passing through A somewhere in the middle of the trip is allowed.*)



19. The lengths of the two legs of a right triangle are in the ratio of 7 : 24. The distance between its incenter and its circumcenter is 1. Find its area. (*Recall that the* ***incenter*** *of a triangle is the center of its inscribed circle and the* ***circumcenter*** *is the center of its circumscribing circle.*)

20. Let $\lfloor x \rfloor$ be the greatest integer not exceeding x. For instance, $\lfloor 3.4 \rfloor = 3$, $\lfloor 2 \rfloor = 2$, and $\lfloor -2.7 \rfloor = -3$. Determine the value of the constant $\lambda > 0$ so that $2\lfloor \lambda n \rfloor = 1 - n + \left\lfloor \lambda \lfloor \lambda n \rfloor \right\rfloor$ for all positive integers n.

PART II. Show your solution to each problem. Each complete and correct answer is worth ten points.

1. Let x and y be real numbers that satisfy the following system of equations:

$$\begin{cases} \dfrac{x}{x^2y^2-1} - \dfrac{1}{x} &= 4 \\ \dfrac{x^2y}{x^2y^2-1} + y &= 2 \end{cases}$$

   Find all possible values of the product xy.

2. Let BH be the altitude from the vertex B to the side AC of an acute-angled triangle ABC. Let D and E be the midpoints of AB and AC, respectively, and F the reflection of H across the line segment ED. Prove that the line BF passes through the circumcenter of ΔABC.

3. A function $g : \mathbb{N} \to \mathbb{N}$ satisfies the following:

   (a) If m is a proper divisor of n, then $g(m) < g(n)$.

   (b) If m and n are relatively prime and greater than 1, then

$$g(mn) = g(m)g(n) + (n+1)g(m) + (m+1)g(n) + m + n.$$

   Find the least possible value of $g(2016)$.

19th Philippine Mathematical Olympiad

Area Stage

19 November 2016

PART I. (3 points each)

1. 504
2. 6048
3. 7.2 units
4. 597/2 or 298.5
5. $2\sqrt{3}$
6. $\dfrac{\pi}{4}$
7. $\dfrac{1}{48}$
8. $\dfrac{15}{2}$
9. 9375
10. $y = -7x + \dfrac{28}{5}$
11. 6
12. 391
13. $\dfrac{48 - 32\sqrt{2}}{9}$
14. 5^{1008}
15. 470
16. $2015 - k$
17. 3
18. 128
19. $\dfrac{336}{325}$
20. $1 + \sqrt{2}$

PART II. (10 points each)

1. The given system can be expressed as follows:

$$\begin{cases} \dfrac{x}{x^2y^2 - 1} - \frac{1}{x} &= 4 \\ \dfrac{x^2y}{x^2y^2 - 1} + y &= 2 \end{cases} \Rightarrow \begin{cases} \dfrac{x^2}{x^2y^2 - 1} - 1 &= 4x \\ \dfrac{x^2}{x^2y^2 - 1} + 1 &= \dfrac{2}{y} \end{cases}$$

We then have

$$4x + \frac{2}{y} = \frac{2x^2}{x^2y^2 - 1} \Rightarrow 2x + \frac{1}{y} = \frac{x^2}{x^2y^2 - 1}$$

and

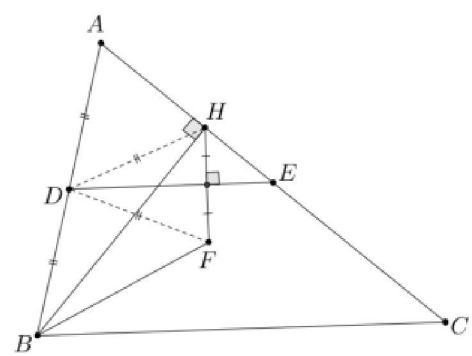
$$4x - \frac{2}{y} = -2 \Rightarrow 2x - \frac{1}{y} = -1,$$

which gives us

$$\begin{aligned}
\left(2x+\frac{1}{y}\right)\left(2x-\frac{1}{y}\right) &= \frac{-x^2}{x^2y^2-1} \\
4x^2-\frac{1}{y^2} &= \frac{-x^2}{x^2y^2-1} \\
\frac{4x^2y^2-1}{y^2} &= \frac{-x^2}{x^2y^2-1} \\
\left(4x^2y^2-1\right)\left(x^2y^2-1\right) &= -x^2y^2 \\
4x^4y^4-5x^2y^2+1 &= -x^2y^2 \\
4x^4y^4-4x^2y^2+1 &= 0 \\
\left(2x^2y^2-1\right)^2 &= 0 \Rightarrow x^2y^2=\frac{1}{2} \Rightarrow \boxed{xy=\pm\frac{1}{\sqrt{2}}}
\end{aligned}$$

2. This problem is taken from the ***2015 Iranian Geometry Olympiad***.

   Solution 1. Let O be the circumcenter of ΔABC. Since $\angle OBA = 90° - \angle C$, it suffices to show that $\angle FBA = 90° - \angle C$.



   Note that $AD = BD = DH$ and $DH = DF$. Therefore, quadrilateral $AHFB$ is cyclic (with circumcenter D), and so $\angle FBA = \angle FHE = 90° - \angle DEH$. Since DE is parallel to BC, $\angle DEH = \angle C$, and $\angle FBA = 90° - \angle C$. □

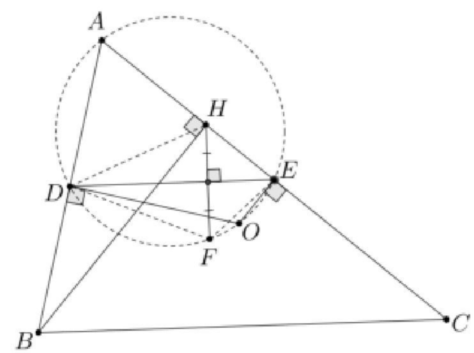
   Solution 2. As before, denote by O the circumcenter of ΔABC. Then the quadrilateral $ADOE$ is cyclic. Also, we know that $AD = HD = DB$, therefore,

$$\angle A = \angle DHA = 180° - \angle DHE = 180° - \angle DFE$$

   Therefore, $ADFE$ is cyclic. Since $ADFOE$ is cyclic, $DFOE$ is also cyclic, and

$$\angle C = \angle DEA = \angle DEF = \angle DOF$$

   On the other hand, $\angle C = \angle DOB$, so $\angle DOF = \angle DOB$, therefore B, F, and O are collinear. □



3. Consider $h(x) := g(x) + x + 1$. We have that, for m, n coprime and greater than 1,

$$\begin{aligned} h(m)h(n) &= (g(m) + m + 1)(g(n) + n + 1) \\ &= g(m)g(n) + (n+1)g(m) + (m+1)g(n) + mn + m + n + 1 \\ &= g(mn) + mn + 1 \\ &= h(mn). \end{aligned}$$

Repeating this, we find that more generally, if $m_1, m_2, \ldots, m_k$ are pairwise coprime positive integers all greater than 1,

$$h\left(\prod_{i=1}^{k} m_i\right) = \prod_{i=1}^{k} m_i.$$

Hence, it suffices to consider h, and thus g, only on prime powers. Since

$$g(p^n) > g(p^{n-1}) > \cdots > g(p) > g(1) \geq 1,$$

we have $g(p^n) \geq n + 1$. Indeed, taking $g(1) = 1$, $g(p^n) = n + 1$ gives us a well-defined function g on $\mathbb{N}$. To solve for $g(2016)$, we solve for $h(2016)$ first, noting that $2016 = 2^5{\cdot}3^2{\cdot}7^1$:

$$\begin{aligned} h(2016) &= h(2^5)h(3^2)h(7^1) \\ &= (7 + 2^5)(4 + 3^2)(3 + 7^1) \\ &= 5070 \end{aligned}$$

and so $g(2016) = 5070 - 2017 = \boxed{3053}$. This is the minimum possible value of $g(2016)$. $\square$

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