20

PHILIPPINE MATHEMATICAL OLYMPIAD

MCMLXXXIV

PMO

NATIONAL STAGE

Ateneo de Manila University

20 January 2018

SCHEDULE

7:00 AM - 7:30 AM	**Registration** PLDT-CTC Room 408
7:30 AM - 12:00 NN	**Written Phase** PLDT-CTC Room 408
12:00 NN - 1:30 PM	**Lunch Break**
1:30 PM - 5:30 PM	**Oral Phase** PLDT-CTC Room 413
5:30 PM - 8:30 PM	**Dinner and Awarding Ceremonies** 5th Floor, First Pacific Hall Rizal Library

ABOUT THE PMO

First held in **1984**, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: **(1)** to awaken greater interest in and promote the appreciation of mathematics among students and teachers; **(2)** to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; **(3)** to provide a vehicle for the professional growth of teachers; and **(4)** to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the **International Mathematical Olympiad (IMO).** It is followed by the **Mathematical Olympiad Summer Camp (MOSC)**, a five-phase program for the twenty two national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the twentieth since 1984. Four thousand six hundred seventy-eight (4678) junior and senior high school students from three hundred eighty-six (386) schools all over the country took the qualifying examination, out of these, two hundred eight (208) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty-one and these twenty-one students will compete for the top three positions and hopefully move on to represent the country in the **59th International Mathematical Olympiad**, which will be held in **Cluj-Napoca, Romania**, from **03 to14 July 2018**.

MESSAGE

Greetings of peace!

The much-anticipated Final Stage of the 20th Philippine Mathematical Olympiad (PMO) provides the perfect start to the year 2018, highlighted by a show of excellence among the country's top young mathematicians.

As the longest running mathematics competition in the archipelago, the PMO has constantly produced some of the current leading minds in the country's mathematics scene, which in turn, helped us achieve milestones in the international arena. Without mentioning names, the alumni of the PMO plum prove to be outstanding professionals in their respective fields here and abroad. This is a testament to the quality of the competition and the aspirations it upholds.

The PMO paved way for the Philippines to share the limelight in the grandest stages in mathematics, particularly, in the International Mathematics Olympiad (IMO). For almost a decade now, the country's ranking continuously surges at an all-time high—currently at number 17 in the world—along with the multiple medals that have been won, none bigger than the two gold medals won in 2016. Clearly, it is through the PMO and the people behind it that we are able to soar high in this ever-growing domain.

Our young students' improving performance drives us, from the Science Education Institute (SEI), to continue supporting competitions like the PMO as a way to develop our future leaders in the field of science, technology, engineering, and mathematics. Together with the Mathematics Society of the Philippines and the schools taking part in this PMO, SEI shall remain as one of the forces sustaining the great momentum towards development.

We wish our participants all the best and hereby express our salutations to all the winners.

JOSETTE T. BIYO

Director

Science Education Institute

Department of Science and Technology

MESSAGE

My warmest greetings and congratulations to the area stage winners and twenty-one national finalists in the 20th Philippine Mathematical Olympiad! With 4,678 participants from 386 schools, this year's PMO has the highest participation since this prestigious mathematics competition started in 1984.

With the successful partnership of the Mathematical Society of the Philippines and the Department of Science and Technology, the PMO has become a more relevant and much-awaited activity for math enthusiasts and has raised national awareness of the mathematical talent and potential of Filipino students. After years of an elusive gold medal in the International Mathematical Olympiad, the Philippines finally got the much desired gold in 2016 and ranked 17th among more than 100 participating countries. In 2017, our team won three silver and three bronze medals, maintaining our country ranking. This brings much pressure to our future IMO teams. I am confident that we will continue performing well in the IMO, with the vast source of talents coming from the PMO, the support of DOST and sponsors, and the guidance of excellent, dedicated trainors.

To the twenty-one finalists, I wish you good luck as you embark on another journey, training for the IMO. I hope the training will be a venue for growing and learning, and that your experience in math competitions will bear a strong positive imprint to guide you in your future endeavors. May you aspire to wield your exceptional talent in mathematics to become excellent mathematicians, scientists, or engineers who will help our country advance in science and technology.

My most heartfelt gratitude to the zealous PMO team headed by Dr. Nerissa M. Abara, the regional coordinators, dedicated coaches, supportive school administrators, generous sponsors, and of course, the nurturing parents. Your contribution to the success of the 20th PMO and our young math wizards is priceless.

The MSP recognizes the importance of the PMO in promoting and enhancing mathematics in our country. It is our commitment to continue conducting the PMO in our effort to discover and nurture mathematical talents.

Mabuhay tayong lahat!

MARIAN P. ROQUE

President

Mathematical Society of the Philippines

MESSAGE

To God be the Glory!

With much dedication, the Philippine Math Olympiad quest has finally geared the Filipino youth towards global excellence by winning our first two gold medals in the recent 2016 International Math Olympiad. Congratulations!

May this New Year 2018 bring more aspirations to leap further, “the greater the sacrifice, the greater the reward.”

SHARP

My deep regards to all the parents who continually give full support for the success of their children.

Success is about each one doing his or her part.

Romans 8:28: *“And we know that all things work together for good to those who love God, to those who are called according to His purpose.”*

LUCERO ONG

Brand Consultant

Sharp Calculators

MESSAGE

My warmest greetings and congratulations to the Area Stage winners and national finalists of the 20th Philippine Mathematical Olympiad (PMO). May your success bring more honor to our country as the PMO begins the next step in the selection process for the national team's participation in the next International Mathematical Olympiad.

The Foundation for Upgrading the Standard of Education (FUSE) shares a common goal with the Mathematical Society of the Philippines (MSP), which is to promote and develop mathematics education in the Philippines.

FUSE is one with MSP and the Department of Science and Technology in their aspirations to further raise the bar in teaching math to Filipino students.

Thank you for creating a venue for high school students wherein they can hone and test their math skills in a friendly competition and hopefully, encourage them to embrace the field of math and the sciences.

Foundation for Upgrading the Standard of Education, Inc.

I would also like to commend the coaches and trainers for nurturing and producing gifted math competitors from high school; the participating schools for actively supporting PMO; and the parents concerned for inspiring and motivating their children.

Good luck to members of Team Philippines who will eventually represent our country in the upcoming International Mathematical Olympiad. I am certain that this year's team will equal, if not better, the country's performance in Rio de Janeiro, Brazil.

A peaceful year ahead.

LUCIO C. TAN
Vice-Chairman, Board of Trustees
Foundation for Upgrading the
Standard of Education Inc.

MESSAGE

C&E Publishing, Inc.
Towards Academic and Professional Excellence

THE PMO TEAM

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Melvin Vidar
Len Patrick Dominic Garces

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Victoria May Paguio

SECRETARY

May Anne Tirado
Gaudella Ruiz

LAYOUT DESIGNER

Rolando Perez III

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Lu Kevin Ong
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Jude Buot

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REGION II
Crizaldy Binarao

REGION III
Eduard Taganap

REGION IV-A
Sharon Lubag

REGION IV-B
Sheila Grace Soriano

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Lorna Almocera

REGION VIII
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REGION IX
Rochelleo Mariano

REGION X, XII, ARMM
Gina Malacas

REGION XI
Jonald Fenecios

REGION XIII
Miraluna Herrera

NCR
Dennis Leyson
Renier Mendoza

AREA STAGE WINNERS

LUZON

1	**Patupat, Albert John**	De La Salle University Integrated School
2	**Torres, Vince Jan**	Santa Rosa Science and Technology High School
3	**Cajayon, Emmanuel Osbert**	Emilio Aguinaldo College

VISAYAS

1	**Yu, Jonathan Condrad**	Philippine Christian Gospel Church
2	**Wee, Makarius**	Philippine Christian Gospel Church
3	**King, William Joshua**	Bethany Christian School

MINDANAO

1	**Ty, Sean Anderson**	Zamboanga Chong Hua High School
2	**Go, Xavier Jefferson Ray**	Zamboanga Chong Hua High School
3	**Lim, Fedrick Lance**	Zamboanga Chong Hua High School
3	**Ty, Stephen James**	Zamboanga Chong Hua High School

NCR

1	**Gonzales, Andres Rico III**	Colegio de San Juan de Letran
2	**Gelera, Christian Philip**	Philippine Science High School – Main Campus
3	**Quines, Carl Joshua**	Valenzuela City School of Mathematics and Science
3	**Reyes, Steven**	Saint Jude Catholic School

PRIZES

The prizes for the TOP THREE for each AREA (Luzon, Visayas, Mindanao, NCR) are:

FIRST PLACE - Medal and SHARP Calculator

SECOND PLACE - Medal and SHARP Calculator

THIRD PLACE - Medal and SHARP Calculator

The prizes for the TOP THREE in the NATIONAL STAGE are:

CHAMPION - P 20,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

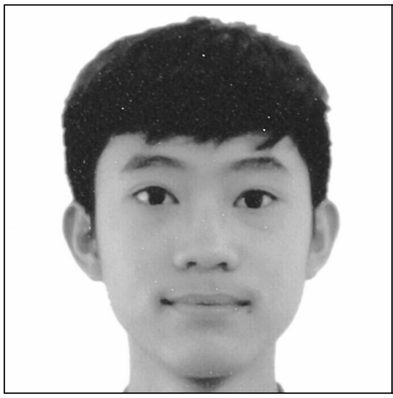
FIRST RUNNER-UP - P 15,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

SECOND RUNNER-UP - P 10,000, Medal, Trophy, and SHARP Calculator with SHARP Goodies

The coaches of the top three will receive P 5,000, P 3,000, and P 2,000, respectively.

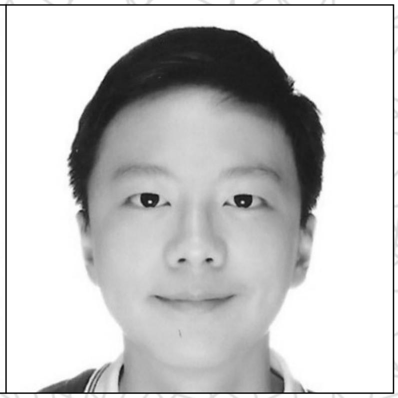
The schools of the top three will receive trophies.

NATIONAL FINALISTS

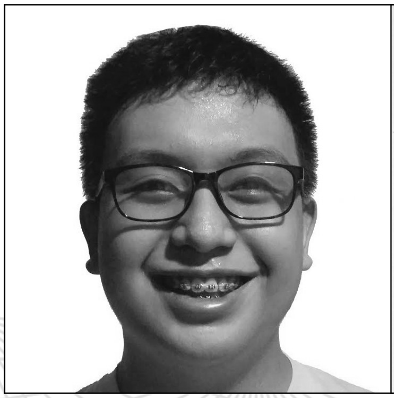


IMMANUEL JOSIAH
BALETE
St. Stephen's High School

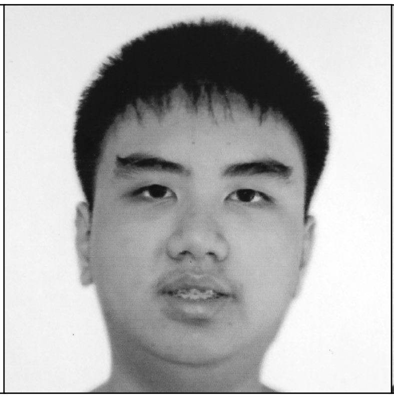
EMMANUEL OSBERT
CAJAYON
Emilio Aguinaldo College



EION NIKOLAI
CHUA
International School Manila

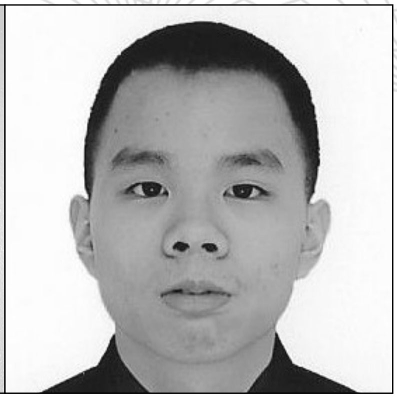


ELIJAMIN WOLFGANG
CLAVERIA
Philippine Science High School - Main Campus

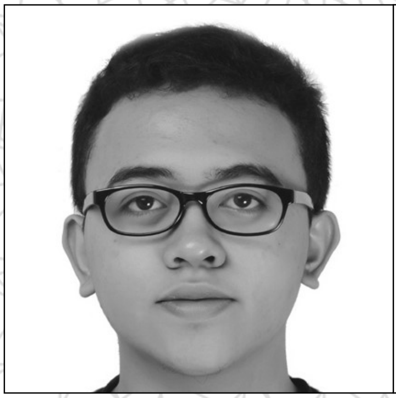


VINCENT
DELA CRUZ
Valenzuela City School of Mathematics and Science

KYLE PATRICK
DULAY
Philippine Science High School - Main Campus



LAWRENCE GABRIEL
DY
CCF Life Academy

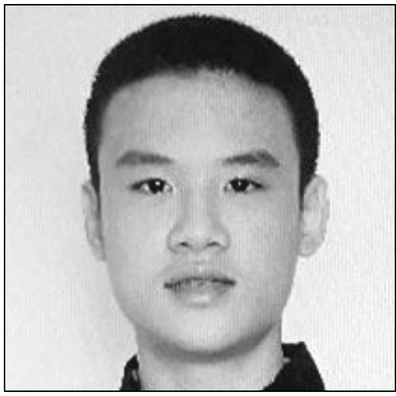


CHRISTIAN PHILIP
GELERA
Philippine Science High School – Main Campus

ANDRES RICO
GONZALES III
Colegio de San Juan de Letran

MATTHEW ANGELO
ISIDRO
Saint Jude Catholic School

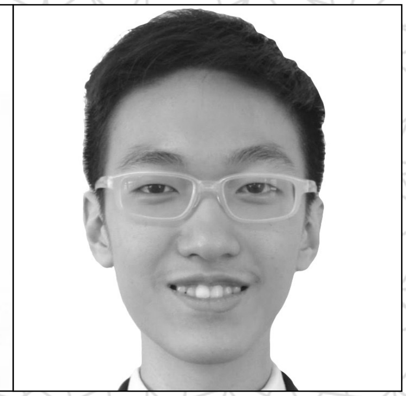
NATIONAL FINALISTS



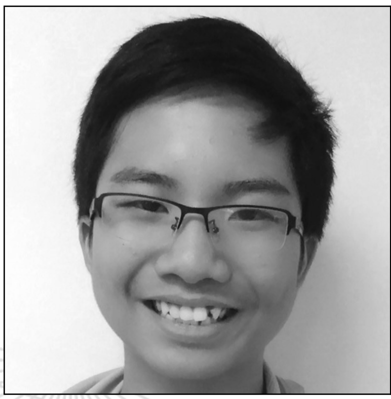
DION STEPHAN
ONG
Ateneo de Manila Junior High School

STEFAN MARCUS
ONG
Saint Jude Catholic School

ALBERT JOHN
PATUPAT
De La Salle University Integrated School

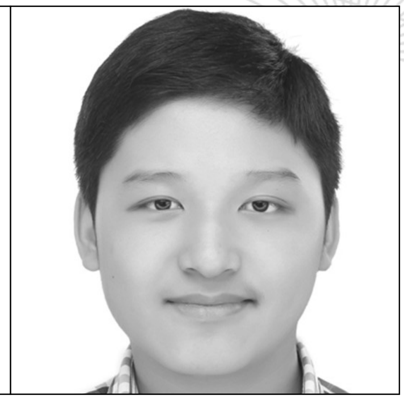


SHAQUILLE WYAN
QUE
Grace Christian College

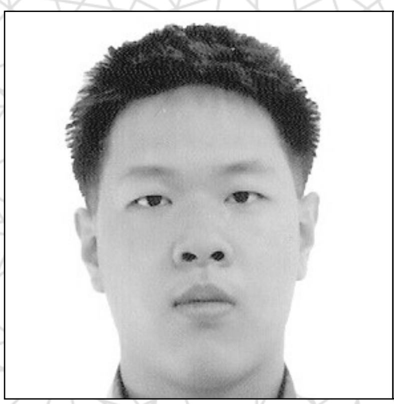


CARL JOSHUA
QUINES
Valenzuela City School of Mathematics and Science

STEVEN
REYES
Saint Jude Catholic School

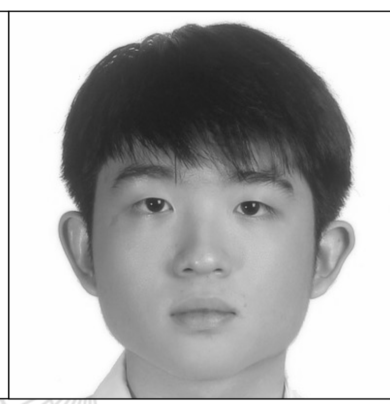


BRYCE AINSLEY
SANCHEZ
Grace Christian College

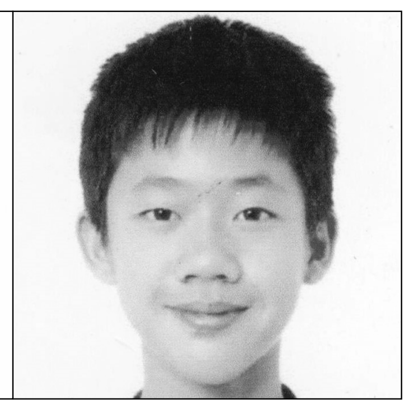


JOSEI
TOLENTINO
Uno High School

VINCE JAN
TORRES
Santa Rosa Science and Technology High School



SEAN ANDERSON
TY
Zamboanga Chong Hua High School



JULIAN
YU
British School Manila

PMO HIGHLIGHTS

PMO HIGHLIGHTS

20th Philippine Mathematical Olympiad

Qualifying Stage, 28 October 2017

A project of the **Mathematical Society of the Philippines (MSP)** and the **Department of Science and Technology - Science Education Institute (DOST-SEI)** in partnership with **HARI Foundation** and **Manulife Business Processing Services**

PART I. Choose the best answer. Each correct answer is worth two points.

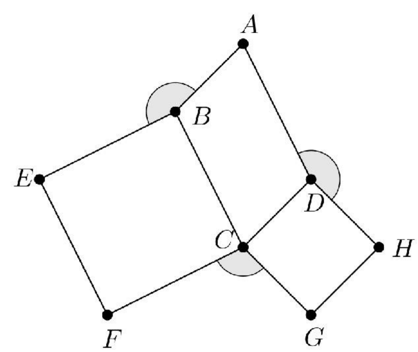
1. Find x if $\frac{79}{125}\left(\frac{79+x}{125+x}\right) = 1$.

   (a) 0 (b) -46 (c) -200 (d) -204

2. The line $2x + ay = 5$ passes through $(-2,-1)$ and $(1,b)$. What is the value of b?

   (a) $-\frac{1}{2}$ (b) $-\frac{1}{3}$ (c) $-\frac{1}{4}$ (d) $-\frac{1}{6}$

3. Let $ABCD$ be a parallelogram. Two squares are constructed from its adjacent sides, as shown in the figure below. If $\angle BAD = 56°$, find $\angle ABE + \angle ADH + \angle FCG$, the sum of the three highlighted angles.



   (a) 348° (b) 384° (c) 416° (d) 432°

4. For how many integers x from 1 to 60, inclusive, is the fraction $\frac{x}{60}$ already in lowest terms?

   (a) 15 (b) 16 (c) 17 (d) 18

5. Let r and s be the roots of the polynomial $3x^2 - 4x + 2$. Which of the following is a polynomial with roots $\frac{r}{s}$ and $\frac{s}{r}$?

   (a) $3x^2 + 2x + 3$ (b) $3x^2 + 2x - 3$ (c) $3x^2 - 2x + 3$ (d) $3x^2 - 2x - 3$

6. If the difference between two numbers is a and the difference between their squares is b, where $a, b > 0$, what is the sum of their squares?

(a) $\frac{a^2+b^2}{a}$ (b) $2\left(\frac{a+b}{a}\right)^2$ (c) $\left(a+\frac{b}{a}\right)^2$ (d) $\frac{a^4+b^2}{2a^2}$

7. Evaluate the sum

$$\sum_{n=3}^{2017} \sin\left(\frac{(n!)\pi}{36}\right).$$

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

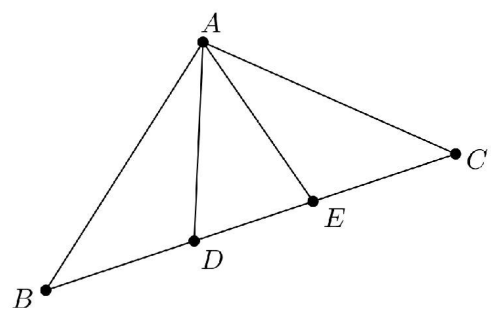
8. In $\triangle ABC$, D is the midpoint of BC. If the sides AB, BC, and CA have lengths $4, 8,$ and 6, respectively, then what is the numerical value of AD^2 ?

(a) 8 (b) 10 (c) 12 (d) 13

9. Let A be a positive integer whose leftmost digit is 5 and let B be the number formed by reversing the digits of A. If A is divisible by 11, 15, 21, and 45, then B is not always divisible by

(a) 11 (b) 15 (c) 21 (d) 45

10. In $\triangle ABC$, the segments AD and AE trisect $\angle BAC$. Moreover, it is also known that $AB = 6, AD = 3, AE = 2.7, AC = 3.8$ and $DE = 1.8$. The length of BC is closest to which of the following?



(a) 8 (b) 8.2 (c) 8.4 (d) 8.6

11. Let $\{a_n\}$ be a sequence of real numbers defined by the recursion $a_{n+2} = a_{n+1} - a_n$ for all positive integers n. If $a_{2013} = 2015$, find the value of $a_{2017} - a_{2019} + a_{2021}$.

(a) 2015 (b) -2015 (c) 4030 (d) -4030

12. A *lattice point* is a point whose coordinates are integers. How many lattice points are strictly inside the triangle formed by the points $(0,0)$, $(0,7)$, and $(8,0)$?

(a) 21 (b) 22 (c) 24 (d) 28

13. Find the sum of the solutions to the logarithmic equation

$$x^{\log x} = 10^{2-3\log x+2(\log x)^2},$$

where $\log x$ is the logarithm of x to the base 10.

(a) 10 (b) 100 (c) 110 (d) 111

14. Triangle ABC has $AB = 10$ and $AC = 14$. A point P is randomly chosen in the interior or on the boundary of triangle ABC. What is probability that P is closer to AB than to AC?

(a) 1/4 (b) 1/3 (c) 5/7 (d) 5/12

15. Suppose that $\{a_n\}$ is a nonconstant arithmetic sequence such that $a_1 = 1$ and the terms a_3, a_{15}, a_{24} form a geometric sequence in that order. Find the smallest index n for which $a_n < 0$.

(a) 50 (b) 51 (c) 52 (d) 53

PART II. Choose the best answer. Each correct answer is worth three points.

1. Two red balls, two blue balls, and two green balls are lined up into a single row. How many ways can you arrange these balls such that no two adjacent balls are of the same color?

(a) 15 (b) 30 (c) 60 (d) 90

2. What is the sum of the last two digits of $403^{(10^{10}+6)}$?

(a) 9 (b) 10 (c) 11 (d) 12

3. How many strictly increasing finite sequences (having one or more terms) of positive integers less than or equal to 2017 with an odd number of terms are there?

(a) 2^{2016} (b) $\dfrac{4034!}{(2017!)^2}$ (c) $2^{2017} - 2017^2$ (d) $2^{2018} - 1$

4. If one of the legs of a right triangle has length 17 and the lengths of the other two sides are integers, then what is the radius of the circle inscribed in that triangle?

(a) 8 (b) 14 (c) 11 (d) 10

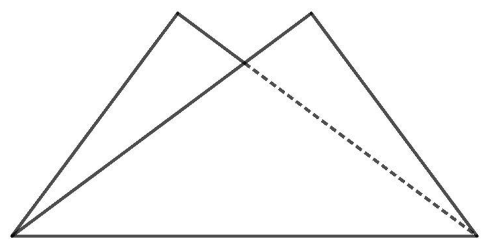
5. Let N be the smallest three-digit positive number with exactly 8 positive even divisors. What is the sum of the digits of N?

(a) 4 (b) 9 (c) 12 (d) 13

6. Let a, b, c be randomly chosen (in order, and with replacement) from the set $\{1, 2, 3, \ldots, 999\}$. If each choice is equally likely, what is the probability that $a^2 + bc$ is divisible by 3?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{7}{27}$ (d) $\frac{8}{27}$

7. Folding a rectangular sheet of paper with length ℓ and width w in half along one of its diagonals, as shown in the figure below, reduces its "visible" area (the area of the pentagon below) by 30%. What is $\frac{\ell}{w}$?



(a) $\frac{4}{3}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\sqrt{5}$ (d) $\frac{\sqrt{5}}{2}$

8. Find the sum of all positive integers k such that $k(k + 15)$ is a perfect square.

(a) 63 (b) 65 (c) 67 (d) 69

9. Let $f(n) = \frac{n}{3^r}$ where n is an integer, and r is the largest nonnegative integer such that n is divisible by 3^r. Find the number of distinct values of $f(n)$ where $1 \leq n \leq 2017$.

(a) 1344 (b) 1345 (c) 1346 (d) 1347

10. If A, B, and C are the angles of a triangle such that

$$5 \sin A + 12 \cos B = 15$$

and

$$12 \sin B + 5 \cos A = 2,$$

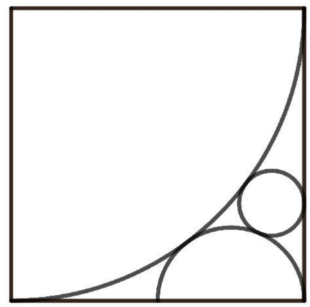
then the measure of angle C is

(a) $150°$ (b) $135°$ (c) $45°$ (d) $30°$

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. How many three-digit numbers are there such that the sum of two of its digits is the largest digit?

2. In the figure, a quarter circle, a semicircle and a circle are mutually tangent inside a square of side length 2. Find the radius of the circle.



3. Find the minimum value of
$$\frac{18}{a+b} + \frac{12}{ab} + 8a + 5b,$$
where a and b are positive real numbers.

4. Suppose $\dfrac{\tan x}{\tan y} = \dfrac{1}{3}$ and $\dfrac{\sin 2x}{\sin 2y} = \dfrac{3}{4}$, where $0 < x, y < \dfrac{\pi}{2}$. What is the value of $\dfrac{\tan 2x}{\tan 2y}$?

5. Find the largest positive real number x such that
$$\frac{2}{x} = \frac{1}{\lfloor x \rfloor} + \frac{1}{\lfloor 2x \rfloor},$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Answers

Part I. (2 points each)

1. D
2. B
3. C
4. B
5. C
6. D
7. B
8. B
9. C
10. A
11. D
12. A
13. C
14. D
15. C

Part II. (3 points each)

1. B
2. C
3. A
4. A
5. B
6. A
7. C
8. C
9. B
10. D

Part III. (6 points each)

1. 279 (or 126)[1]
2. $\frac{2}{9}$
3. 30
4. $-\frac{3}{11}$
5. $\frac{20}{7}$

[1] We are also accepting the answer 126, as the wording of the problem seems to suggest that the sum of the two digits is equal to the third digit.

20th Philippine Mathematical Olympiad

Area Stage, 25 November 2017

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. Suppose that x and y are nonzero real numbers such that $\left(x+\frac{1}{y}\right)\left(y+\frac{1}{x}\right)=7$. Find the value of $\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{x^2}\right)$.

2. In how many ways can the integers

$$-5,-4,-3,-2,-1,1,2,3,4,5$$

be arranged in a circle such that the product of each pair of adjacent integers is negative? (Assume that arrangements which can be obtained by rotation are considered the same.)

3. Let P be a point inside the isosceles trapezoid $ABCD$ where AD is one of the bases, and let PA, PB, PC, and PD bisect angles A, B, C, and D respectively. If $PA=3$ and $\angle APD=120^\circ$, find the area of trapezoid $ABCD$.

4. Determine the number of ordered pairs of integers (p,q) for which $p^2+q^2<10$ and $-2^p\le q\le 2^p$.

5. Let $f(x)=\sqrt{4\sin^4 x-\sin^2 x\cos^2 x+4\cos^4 x}$ for any $x\in\mathbb{R}$. Let M and m be the maximum and minimum values of f, respectively. Find the product of M and m.

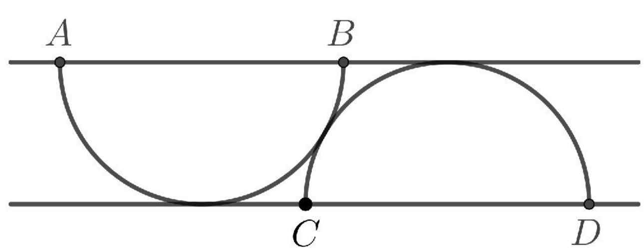
6. A semicircle Γ has diameter $AB=25$. Point P lies on AB with $AP=16$ and C is on the semicircle such that $PC\perp AB$. A circle ω is drawn so that it is tangent to segment PC, segment PB, and Γ. What is the radius of ω?

7. Determine the area of the polygon formed by the ordered pairs (x,y) where x and y are positive integers which satisfy the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{13}.$$

8. Let A be the sum of the decimal digits of the largest 2017-digit multiple of 7 and let B be the sum of the decimal digits of the smallest 2017-digit multiple of 7. Find $A-B$.

9. Two semicircles, each with radius $\sqrt{2}$, are tangent to each other, as shown in the figure below. If $AB\parallel CD$, determine the length of segment AD.



10. The boat is sinking! Passengers must then be saved, but the rescuer must know their count. If the passengers group themselves into 7, one group will only have 4 passengers. If the passengers group themselves into 11, one group will only have 7 passengers. If the passengers group themselves into 13, one group will only have 10 passengers. How many passengers are there if the boat carried at most 1000 passengers?

11. Given $a_n \in \mathbb{Z}$ with $a_{10} = 11$ and $a_9 = -143$, determine the number of polynomials of the form

$$P(x) = \sum_{n=0}^{10} a_n x^n$$

such that the zeros of $P(x)$ are all positive integers.

12. In how many ways can nine chips be selected from a bag that contains three red chips, three blue chips, three white chips, and three yellow chips? (Assume that the order of selection is irrelevant and that the chips are identical except for their color.)

13. Let L_1 be the line with equation $6x - y + 6 = 0$. Let P and Q be the points of intersection of L_1 with the x-axis and y-axis, respectively. A line L_2 that passes through the point $(1, 0)$ intersects the y-axis and L_1 at R and S, respectively. If O denotes the origin and the area of ΔOPQ is six times the area of ΔQRS, find all possible equations of the line L_2. Express your answer in the form $y = mx + b$.

14. Find the smallest positive integer whose cube ends in 2017.

15. Let $\{x_k\}_{k=1}^{n}$ be a sequence whose terms come from $\{2, 3, 6\}$. If

$$x_1 + x_2 + \cdots + x_n = 633 \quad \text{and} \quad \frac{1}{x_1^2} + \frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} = \frac{2017}{36},$$

find the value of n.

16. Let S be a subset of $\{1, 2, \ldots, 2017\}$ such that no two elements of S have a sum divisible by 37. Find the maximum number of elements that S can have.

17. In cyclic pentagon $ABCDE$, $\angle ABD = 90°, BC = CD$, and AE is parallel to BC. If $AB = 8$ and $BD = 6$, find AE^2.

18. The edges of a square are to be colored either red, blue, yellow, pink, or black. Each side of the square can only have one color, but a color may color many sides. How many different ways are there to color the square if two ways that can be obtained from each other by rotation are identical?

19. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If $a_n \lfloor a_n \rfloor = 49^n + 2n + 1$, find the value of $2S + 1$, where $S = \left\lfloor \sum_{n=1}^{2017} \frac{a_n}{2} \right\rfloor$.

20. A spider and a fly are on diametrically opposite vertices of a web in the shape of a regular hexagon. The fly is stuck and cannot move. On the other hand, the spider can walk freely along the edges of the hexagon. Each time the spider reaches a vertex, it randomly chooses between two adjacent edges with equal probability, and proceeds to walk along that edge. On average, how many edge lengths will the spider walk before getting to the fly?

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Find all pairs (r, s) of real numbers such that the zeros of the polynomials

$$f(x) = x^2 - 2rx + r$$

and

$$g(x) = 27x^3 - 27rx^2 + sx - r^6$$

are all real and nonnegative.

2. A point P is chosen randomly inside the triangle with sides 13, 20, and 21. Find the probability that the circle centered at P with radius 1 will intersect at least one of the sides of the triangle.

3. Define a sequence of integers as follows: $a_1 = 1, a_2 = 2$, and for $k \in \mathbb{N}$, $a_{k+2} = a_{k+1} + a_k$. How many different ways are there to write 2017 as a sum of distinct elements of this sequence?

Answers to the 20th PMO Area Stage

Part I. (3 points each)

1. 25
2. 2880
3. $6\sqrt{3}$
4. 17
5. $\sqrt{7}$
6. 4
7. 12096
8. 18144
9. $2(1+\sqrt{3})$
10. 634
11. 3
12. 20
13. $y=-3x+3$, $y=-10x+10$
14. 9073
15. 262
16. 991
17. 338/5
18. 165
19. $\dfrac{7(7^{2017}-1)}{6}$
20. 9

Part II. (10 points each, full solutions required)

1. Find all pairs (r,s) of real numbers such that the zeros of the polynomials

$$f(x)=x^2-2rx+r$$

and

$$g(x)=27x^3-27rx^2+sx-r^6$$

are all real and nonnegative.

Answers: $(0,0)$, $(1,9)$

Solution:

Let x_1,x_2 be the zeros of $f(x)$, and let y_1,y_2,y_3 be the zeros of $g(x)$.

By Viete's relation,

$$\begin{aligned} x_1+x_2 &= 2r \\ x_1x_2 &= r \end{aligned}$$

and

$$\begin{aligned} y_1+y_2+y_3 &= r \\ y_1y_2+y_2y_3+y_3y_1 &= \frac{s}{27} \\ y_1y_2y_3 &= \frac{r^6}{27} \end{aligned}$$

Note that

$$\left(\frac{x_1+x_2}{2}\right)^2 \geq x_1x_2 \quad \Rightarrow \quad r^2 \geq r$$

$$\begin{aligned}
\frac{y_1+y_2+y_3}{3} &\geq \sqrt[3]{y_1y_2y_3} \\
\frac{r}{3} &\geq \sqrt[3]{\frac{r^6}{27}} \\
r &\geq r^2
\end{aligned}$$

Hence $r = r^2$, and consequently $x_1 = x_2$ and $y_1 = y_2 = y_3$. Moreover, $r = 0, 1$.

- If $r = 0$, then $f(x) = x^2$ with $x_1 = x_2 = 0$. And since $y_1 = y_2 = y_3$ with $y_1 + y_2 + y_3 = 0$, then ultimately $s = 0$.
- If $r = 1$, then $f(x) = x^2 - 2x + 1 = (x-1)^2$ with $x_1 = x_2 = 1$. And since $y_1 = y_2 = y_3$ with $y_1 + y_2 + y_3 = 1$ then $y_1 = y_2 = y_3 = \frac{1}{3}$. Therefore $s = 9$.

Thus, the possible ordered pairs (r, s) are $\boxed{(0,0)}$ and $\boxed{(1,9)}$.

2. A point P is chosen randomly inside the triangle with sides 13, 20, and 21. Find the probability that the circle centered at P with radius 1 will intersect at least one of the sides of the triangle.

   Answer: 75/196

   Solution 1: Let ABC be a triangle with sides $BC = 13, CA = 20$ and $AB = 21$ and let S be the set of points P such that the circle ω with radius 1 centered at P intersects at least one of the sides of ABC. For a fixed side of ABC (say ℓ), ω intersects ℓ if and only if P lies within one unit from ℓ. This suggests we construct a triangle $A_1B_1C_1$ such that $A_1B_1 \parallel AB, B_1C_1 \parallel BC, C_1A_1 \parallel CA$ and the corresponding parallel sides of $A_1B_1C_1$ and ABC have distance 1. Thus, the set S of such points P forms a region $\mathcal{R}$ outside $A_1B_1C_1$ but inside ABC and the probability is then the ratio of the areas of $\mathcal{R}$ and ABC.

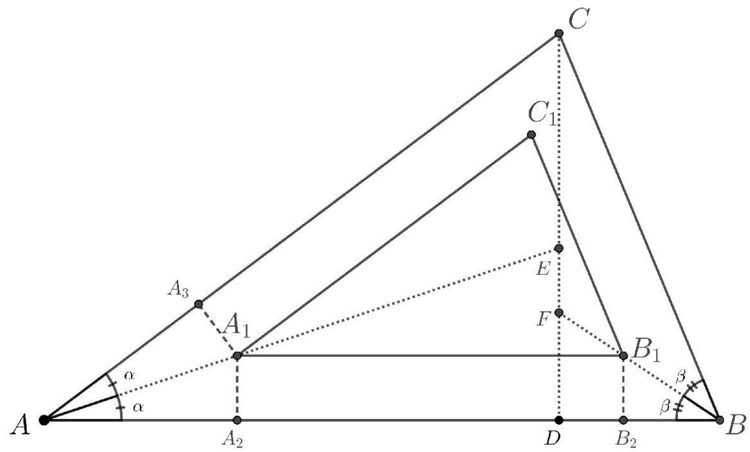
   Observe that ABC and $A_1B_1C_1$ are similar and hence the ratio of their corresponding sides is constant, say $k > 0$. Also, triangle ABC is divided into four regions: the triangle $A_1B_1C_1$ and three trapezoids A_1B_1BA, B_1C_1CB and C_1A_1AC. The region $\mathcal{R}$ then comprises these trapezoids. We use $[\mathcal{P}]$ to denote the area of region $\mathcal{P}$. Using Heron's formula with semiperimeter $s = 27$, we see that $[ABC] = \sqrt{27(27-13)(27-20)(27-21)} = 126$. As ABC and $A_1B_1C_1$ are similar, $[A_1B_1C_1] : [ABC] = k^2$ and with $B_1C_1 = 13k, C_1A_1 = 20k, A_1B_1 = 21k$, we obtain

$$\begin{aligned}
[ABC] &= [A_1B_1C_1] + [A_1B_1BA] + [B_1C_1CB] + [C_1A_1AC] \\
126 &= 126k^2 + \frac{1}{2}(21k+21) + \frac{1}{2}(20k+20) + \frac{1}{2}(13k+13) \\
&= 126k^2 + 27k + 27
\end{aligned}$$

   so $126k^2 + 27k - 99 = 9(k+1)(14k-11) = 0$ and $k = \frac{11}{14}$. Therefore, the probability is

$$\frac{[\mathcal{R}]}{[ABC]} = 1 - \frac{[A_1B_1C_1]}{[ABC]} = 1 - k^2 = 1 - \frac{121}{196} = \frac{75}{196}.$$

Solution 2: The additional points in the figure below (not drawn to scale) are precisely what they appear to be.



We can determine the proportionality constant k between $\triangle A_1B_1C_1$ and $\triangle ABC$ by determining $A_1B_1 = A_2B_2 = 21 - AA_2 - BB_2$. Since $\triangle AA_1A_2$ and $\triangle AA_1A_3$ are congruent right triangles, then AA_1 bisects $\angle A$. Let $\alpha = \angle A_1AA_2$. Then $\tan\alpha = \frac{A_1A_2}{AA_2} = \frac{1}{AA_2}$ so $AA_2 = \cot\alpha$.

From Heron's Formula, $[ABC] = 126$. Since $126 = \frac{1}{2}(21)(CD)$, then $CD = 12$. By the Pythagorean Theorem, $AD = 16$ and $BD = 5$.

Let the bisector of $\angle A$ meet the altitude CD at E. Thus $\frac{DE}{CE} = \frac{AD}{AC} = \frac{16}{20} = \frac{4}{5}$. Since $CE + DE = 12$, then $DE = \frac{16}{3}$. This implies $\tan\alpha = \frac{DE}{AD} = \frac{16/3}{16} = \frac{1}{3}$, and so $AA_2 = \cot\alpha = 3$.

Similarly, BB_1 bisects $\angle B$. If $\beta = \angle B_1BB_2$, then $BB_2 = \cot\beta$. Extend BB_1 to meet CD at F. Since $\frac{DF}{CF} = \frac{BD}{BC} = \frac{5}{13}$ and $CF + DF = 12$, then $DF = \frac{10}{3}$. This implies $\tan\beta = \frac{DF}{BD} = \frac{10/3}{5} = \frac{2}{3}$, and so $BB_2 = \cot\beta = 1.5$.

Finally, $A_2B_2 = 21 - 3 - 1.5 = 16.5$. Thus, $k = \frac{A_1B_1}{AB} = \frac{16.5}{21} = \frac{11}{14}$, and so

$$\frac{[ABC] - [A_1B_1C_1]}{[ABC]} = \frac{[ABC] - k^2[ABC]}{[ABC]} = 1 - k^2 = \frac{75}{196}.$$

3. Define a sequence of integers as follows: $a_1 = 1, a_2 = 2$, and for $k \in \mathbb{N}$, $a_{k+2} = a_{k+1} + a_k$. How many different ways are there to write 2017 as a sum of distinct elements of this sequence?

   Answer: 24

   Solution: Note that these a_ks are in fact the Fibonacci numbers. Denote by $f(n)$ the number of distinct ways to express a number as a sum of a_ks. Note that $2017 = 1597 + 377 + 34 + 8 + 1 = a_{15} + a_{12} + a_8 + a_5 + a_1$.

   We prove the following lemma:

$$a_1 + a_2 + \cdots + a_k = a_{k+2} - 2$$

   This follows simply from induction. It is true for $k = 1$; adding a_{k+1} to both sides and using the fact that $a_{k+1} + a_{k+2} = a_{k+3}$ gives the result.

Now, denote by $f(n)$ the number of ways to express n as a sum of distinct a_ks; we are looking for $f(2017)$. Now, note that any such sum must contain either 1597 or 987. If the sum does not contain 1597, it must certainly contain 987; otherwise, from the lemma, the sum would be at most $1 + 2 + \cdots + 610 = 1595$. Moreover, if the sum contains 987 (but not 1597), it must also contain 610; otherwise, it will be at most $987 + (1 + 2 + \cdots + 377) = 987 + 985 = 1972$.

Hence, $f(2017) = 2f(420)$.

By a similar argument, any sum of 420 must contain either 377 or 233. However, this time, it is perfectly possible for this sum to contain 233 but not 144, since $233 + (1 + 2 + 3 + \cdots + 89) = 464 > 420$. We thus have two cases to deal with.

Case 1: If the sum contains 377, then we have to compute $f(43)$. Now, note that as in the argument from earlier, any sum adding up to 43 contains either 34 or 13+21. Hence $f(43) = 2f(9)$. Repeating this argument, we get $f(9) = 2f(1) = 2$. This gives us $f(43) = 4$.

Case 2: The sum does not contain 377. In this case, the sum must contain 233. We then have to compute $f(187)$. Any sum adding up to 187 must contain either 144 or 89; moreover, if it contains 89, it must contain 55 as well. Hence, $f(187) = 2f(43) = 8$ by our previous computation.

Thus, $f(420) = 3f(43) = 12$, and $f(2017) = 24$.

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Fully aware that poverty is more than just the lack of resources, but of options to pursue a better life, HFI has streamlined its efforts to a more targeted, long-term multi-disciplinary, and multi-stakeholder solution. Education with focus on environmental stewardship is at the core of HFI's contribution to the Philippine agenda for sustainability.

Hope for a nation. Coming to grips with the negative impact of climate change. The Hyundai New Thinkers Circuit (HNTC) scholars and HFI Board of Trustees (standing, center from L-R) Chairman Richard L. Lee, President, Ma. Fe Perez-Agudo, and Member Edward S. Go are up for the challenge.

Says HFI President Ma. Fe Perez-Agudo, "I have always been a believer in empowering people through education. HFI's concept of education is wholistic and integrated with special focus on responding to a clear and present need to build a climate-change resilient Philippines."

With mounting pressure on the environment, HFI launched its flagship program, the Hyundai New Thinkers Circuit (HNTC), in 2013. HNTC is designed to be a premier climate science literacy program that fosters and nurtures the innovative spirit among outstanding public high school students to take the lead in helping build a climate change-resilient Philippines. HNTC has yielded nine scholars who are pursuing studies in the sciences at the country's top universities.

Arabelle Robles is now a junior taking up Agriculture at UP Los Baños. "Most young people nowadays are not interested in pursuing agriculture even if there is a high demand for agriculturists due to low food security to support our growing population," she says.

Bret Michaels de Leon, on his third year in BS Environmental Science at the Ateneo de Manila University considers the HNTC scholarship as one of the best things that ever happened to him. "HFI inspires me to continue doing better in my studies. I do not want to waste the opportunity. One thing is for sure: I will work for Nature."

While Patrick Angelo Narciso of the University of the Philippines Diliman affirms, "The scholarship drives me to study harder and to pursue my dreams. I'm sure I will be helping other people through science."

HFI aims to step up its efforts by entering into high-level regional partnerships, engaging like-minded individuals and institutions to make its dream of inclusive sustainability for every Filipino come true.

Ms. Agudo furthered, "Good corporate citizenship not only spells good business. By empowering people, we generate potential new markets for our products, and potential new manpower and enterprises for the business supply chain. If the cycle never stops, we can give rise to generations of people capable of advancing sustainability in all important aspects of our lives as Filipinos."

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