Qualifying Round

Part I. Each correct answer is worth two points.

1. Let p and q be the roots of $2x^2 - 5x + 1 = 0$. Find the value of $\log_2 p + \log_2 q$.

   (a) 2 (b) 0 (c) 1 (d) -1

2. Let f be a quadratic function of x. If $2y$ is a root of $f(x-y)$, and $3y$ is a root of $f(x+y)$, what is the product of the roots of $f(x)$?

   (a) $6y^2$ (b) $5y^2$ (c) $4y^2$ (d) $3y^2$

3. If $4 + 12 \cdot 4^x = 16 \cdot 16^x$, what is the value of $2^{2x+4} - 2^{2x}$?

   (a) 120 (b) 60 (c) 30 (d) 15

4. Let $r = \log 50$ and $s = \log 80$. Express $7 \log 20$ in terms of r and s.

   (a) $2r + s$ (b) $2r + 3s$ (c) $r + 2s$ (d) $3r + 2s$

5. Determine the slopes of the lines passing through P$(3, 0)$ that intersect the parabola with equation $y = 8 - x^2$ at exactly one point.

   (a) $-4, -8$ (b) $-3, -2$ (c) $-5, -7$ (d) $-4, -7$

6. If $\dfrac{b}{x^3} + \dfrac{1}{x^2} + \dfrac{1}{x} + 1 = 0$, what is $x^3 + x^2 + x + a$?

   (a) ab (b) $a + b$ (c) $b - a$ (d) $a - b$

7. How many triangles can be formed if two sides have lengths 15 and 19 and the third side has even length?

   (a) 13 (b) 14 (c) 15 (d) 16

8. Solve for c in the following system of equations:

$$\begin{aligned} 16^{a+b} &= \tfrac{\sqrt{2}}{2} \\ 16^{b+c} &= 4 \\ 16^{a+c} &= 2\sqrt{2} \end{aligned}$$

   (a) 0 (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{1}{2}$

9. Two die are made so that the chances of getting an even sum is twice that of getting an odd sum. What is the probability of getting an odd sum in a single roll of these two die?

   (a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

10. If $\frac{\log x}{\log y} = 500$, what is the value of $\frac{\log(y/x)}{\log y}$?

   (a) -498 (b) -501 (c) -502 (d) -499

11. There are 5 shmacks in 2 shicks, 3 shicks in 5 shures, and 2 shures in 9 shneids. How many shmacks are there in 6 shneids?

    (a) 5 (b) 8 (c) 2 (d) 1

12. Find the sum of the digits of the integer $10^{1001} - 9$.

    (a) 9010 (b) 9001 (c) 9100 (d) 9009

13. What is the constant term in the expansion of $\left(2x^2 + \frac{1}{4x}\right)^6$?

    (a) $\frac{15}{32}$ (b) $\frac{12}{25}$ (c) $\frac{25}{42}$ (d) $\frac{15}{64}$

14. A square with an area of 40m^2 is inscribed in a semicircle. The area of the square that could be inscribed in the circle with the same radius is

    (a) 100m^2 (b) 120m^2 (c) 80m^2 (d) 140m^2

15. What is the units digit of $25^{2010} - 3^{2012}$?

    (a) 8 (b) 6 (c) 2 (d) 4

Part II. Each correct answer is worth three points.

1. Find the area of the region bounded by the graph of $2x^2 - 4x - xy + 2y = 0$ and the x-axis.

   (a) 9 (b) 12 (c) 4 (d) 6

2. Find all negative solutions to the equation $x = \sqrt[3]{20 + 21\sqrt[3]{20 + 21\sqrt[3]{20 + 21x}}}$

   (a) $-1, -2$ (b) $-5, -3$ (c) $-2, -4$ (d) $-4, -1$

3. Find the sum of the largest and smallest possible values of $9\cos^4 x + 12\sin^2 x - 4$.

   (a) 10 (b) 11 (c) 12 (d) 13

4. Exactly one of the following people is lying. Determine the liar.

   Bee said, “Cee is certainly not a liar.”

   Cee said, “I know Gee is lying.”

   Dee said, “Bee is telling the truth.”

   Gee said, “Dee is not telling the truth.”

   (a) Bee (b) Cee (c) Dee (d) Gee

5. What is the last digit of $2! + 4! + 6! + \ldots + 2010! + 2012!$?

   (a) 6 (b) 7 (c) 8 (d) 9

6. A point (x, y) is called a lattice point if x and y are integers. How many lattice points are there inside the circle of radius $2\sqrt{2}$ with center at the origin?

   (a) 25 (b) 21 (c) 17 (d) 19

7. Find the least possible value of $|x-1|+|x-3|+|x-5|$.

   (a) 2 (b) 4 (c) 1 (d) 3

8. Let JOHN be a rhombus with JH $=16$ and ON $=12$. Let G and P be points in JN and HN respectively such that JG : GN : NP $= 2:2:1$. What is the length GP?

   (a) $\frac{2\sqrt{19}}{5}$ 2.15cm (b) $\frac{2\sqrt{13}}{3}$ (c) $\frac{3\sqrt{17}}{2}$ (d) $\frac{3\sqrt{15}}{2}$

9. If $\sqrt{4+x}+\sqrt{10-x}=6$, find the product $\sqrt{4+x}\sqrt{10-x}$.

   (a) 13 (b) 7 (c) 17 (d) 11

10. The remainders when the polynomial $p(x)$ is divided by $(x+1)$ and $(x-1)$ are 7 and 5, respectively. Find the sum of the coefficients of the odd powers of x.

    (a) -4 (b) 2 (c) -1 (d) 12

Part III. Each correct answer is worth six points.

1. Let $w^3 = 1$. What is a value of $(1+w-w^2)^3+(1-w+w^2)^3$?

   (a) -16 (b) -21 (c) 18 (d) 15

2. How many positive integer pairs x, y satisfy $\sqrt{x}+\sqrt{y}=\sqrt{600}$?

   (a) 8 (b) 7 (c) 6 (d) 5

3. Evaluate $\cos\frac{\pi}{10}+\cos\frac{2\pi}{10}+\cos\frac{3\pi}{10}+\ldots+\cos\frac{19\pi}{10}$.

   (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) -1 (d) 1

4. How many perfect squares divide the number $2!5!6!$?

   (a) 18 (b) 15 (c) 20 (d) 25

5. Each of 12 students has a ticket to one of 12 chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one to the left or to the right of the specified chair?

   (a) 233 (b) 225 (c) 187 (d) 252

Area Stage

Part I. Each correct answer is worth three points. No solution is needed.

1. Let $ABCD$ be a rectangle with $AB = 3$ and $BC = 1$. Let E be the midpoint of AB and let F be a point on the extension of DE such that $|CF| = |CD|$. Find the area of $\triangle CDF$.

2. Solve for all real numbers x satisfying
$$x + \sqrt{x-1} + \sqrt{x+1} + \sqrt{x^2-1} = 4.$$

3. The length of a leg of a right triangle is 5 while the length of the altitude to its hypotenuse is 4. Find the length of the other leg.

4. Find all positive values of a for which the equation $x^2 - ax + 1 = 0$ has roots that differ by 1.

5. Let $RALP$ be a trapezoid with $RA||LP$. Let H be the intersection of its diagonals. If the area of $\triangle RAH$ is 9 and the of $\triangle LPH$ is 16, find the area of the trapezoid.

6. The polynomial function $p(x)$ has the form $x^{10} - 4x^9 + \ldots + ax + k$ where $a, k \in \mathbb{R}$. If $p(x)$ has integral zeros, find the minimum possible positive value of k.

7. How many squares are determined by the lines with equations $x = k^2$ and $y = l^2$, where $k, l \in \{0, 1, 2, 3, \ldots, 9\}$?

8. What is the sum of the first 800 terms of $3, 4, 4, 5, 5, 5, 6, 6, 6, 6, \ldots$?

9. Placed on a really long table are 2011 boxes each containing a number of balls. The 1st and the 2nd box together contain 15 balls, the 2nd and the 3rd box together contain 16 balls, the 3rd and the 4th box together contain 17 balls, and so on. If the first and the last box together contain a total of 1023 balls, how many balls are contained in the last box?

10. Evaluate
$$\sqrt[1000]{1000^{1000} + \binom{1000}{1}1000^{998} + \binom{1000}{2}1000^{996} + \cdots + \binom{1000}{999}1000^{-998} + 1000^{-1000}}.$$

11. Find all ordered pairs (m, n) of integers such that $4^m - 4^n = 255$.

12. Find all ordered pairs (x, y) satisfying the system
$$\begin{aligned} x^2 + 4y^2 - xy &= 10 \\ 2x - 4y + 3xy &= 11. \end{aligned}$$

13. Find the exact value of $\dfrac{\sqrt{3}}{\sin 20^\circ} - \dfrac{1}{\cos 20^\circ}$.

14. There are two values of r such that $x^4 - x^3 - 18x^2 + 52x + k$ has $x - r$ as a factor. If one of them is $r = 2$, what is the other value of r?

15. For what values of k will the system below have no solution?

$$\begin{aligned} (k-3)x + 2y &= k^2 - 1 \\ x + \left(\frac{k-4}{3}\right) y &= 0 \end{aligned}$$

16. Find all positive integers n such that $n^2 - n + 1$ is a multiple of $5n - 4$.

17. If $x \neq y$ and $\dfrac{x}{y} + x = \dfrac{y}{x} + y$, find the sum $\dfrac{1}{x} + \dfrac{1}{y}$.

18. Let DAN be a triangle whose vertices lie on a circle C. Let AE be the angle bisector of $\angle DAN$ with E on C. If $DA = 2, AN = 1, AE = 2.5$, and AE intersects DN at I, find AI.

19. The length d of a tangent, drawn from a point A to a circle, is $\frac{4}{3}$ of the radius r. What is the shortest distance from A to the circle?

20. If $(x-a)(x-b)(x-c)(x-d) = 9$ is solved by $x = 2$, and $a, b, c,$ and d are distinct integers, find the sum $a + b + c + d$.

Part II. Show your solution for each item. Each item is worth ten points.

1. In rectangle $ABCD$, E and F are chosen on $\overline{AB}$ and $\overline{CD}$, respectively, so that $AEFD$ is a square. If $\dfrac{AB}{BE} = \dfrac{BE}{BC}$, determine the value of $\dfrac{AB}{BC}$.

2. Find the integer m so that

$$10^m < \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \frac{99}{100} < 10^{m+1}$$

3. If f is a function such that $f(a+b) = \dfrac{1}{f(a)} + \dfrac{1}{f(b)}$, find all possible values of $f(2011)$.

Answers and Solutions

Qualifying Round

I.	1. D		9. C	II.	1. C	III.	1. A
	2. C		10. D		2. D		2. D
	3. D		11. C		3. C		3. C
	4. B		12. B		4. D		4. C
	5. A		13. D		5. A		5. A
	6. D		14. A		6. A		
	7. B		15. D		7. B		
	8. D				8. C		
					9. D		
					10. C		

Area Stage

I.	1. $\frac{54}{13}$	6. 3	11. $(4,0)$	16. $5,1$
	2. $\frac{5}{4}$	7. 59	12. $(3,1)$, $\left(-2,-\frac{2}{3}\right)$	17. -1
	3. $\frac{20}{3}$	8. 22940	13. 4	18. $\frac{4}{5}$
	4. $\sqrt{5}$	9. 1014	14. -5	19. $\frac{2}{3}r$
	5. 49	10. $\frac{1000001}{1000}$	15. 6	20. 8

II. 1. Let x be BE and y be AE. Note that $AEFD$ is a square so $AE = BC = y$. Also, $AB = BE + AE$ so $AB = x + y$. Since $\dfrac{AB}{BE} = \dfrac{BE}{BC}$ then $\dfrac{x+y}{x} = \dfrac{x}{y}$. Thus, we have, $xy+y^2 = x^2$ which yields to $x^2-xy-y^2 = 0$. Solving for x using the quadratic formula gives us, $x = \dfrac{y \pm \sqrt{y^2-4(1)(-y^2)}}{2} = \left(\dfrac{1\pm\sqrt{5}}{2}\right)y$. However, we will only take $x = \left(\dfrac{1+\sqrt{5}}{2}\right)y$ since the other solution will mean that $x < 0$ which is absurd since x is a measure of length. Thus, $\dfrac{AB}{BC} = \dfrac{x+y}{y} = \dfrac{\left(\dfrac{1+\sqrt{5}}{2}\right)y+y}{y} = \dfrac{1+\sqrt{5}+2}{2} = \dfrac{3+\sqrt{5}}{2}$.

Therefore, the answer is $\boxed{\dfrac{3+\sqrt{5}}{2}}$.

2. Let $a = \dfrac{1}{2}\times\dfrac{3}{4}\times\dfrac{5}{6}\times\cdots\times\dfrac{99}{100} = \dfrac{3}{2}\times\dfrac{5}{4}\times\dfrac{7}{6}\times\cdots\times\dfrac{99}{98}\times\dfrac{1}{100}$. Hence, $a > \dfrac{1}{100} = 10^{-2}$. Thus, $m \geq -2$.

Now, let $b = \dfrac{2}{3}\times\dfrac{4}{5}\times\dfrac{6}{7}\times\cdots\times\dfrac{96}{97}\times\dfrac{98}{99}$. Notice that $\dfrac{2}{3} > \dfrac{1}{2}$, $\dfrac{4}{5} > \dfrac{3}{4}$, $\ldots$, $\dfrac{98}{99} > \dfrac{97}{98}$. Also, since $\dfrac{99}{100} < 1$, we have $a < b$. Since $a > 0$ then $a^2 < ab$. But $ab = \dfrac{1}{100}$, so that $a^2 < \dfrac{1}{100}$. Hence, $a < \dfrac{1}{10} = 10^{-1}$. Thus, $m \leq -2$. Therefore, $\boxed{m=-2}$.

3. Consider $f(0) = f(0+0) = \dfrac{1}{f(0)} + \dfrac{1}{f(0)}$ which gives $[f(0)]^2 = 2$. Thus, $f(0) = \pm\sqrt{2}$.
Let $x = f(2011)$.
If $f(0) = \sqrt{2}$ then $x = f(2011) = f(2011+0) = \dfrac{1}{f(2011)} + \dfrac{1}{f(0)} = \dfrac{1}{x} + \dfrac{1}{\sqrt{2}}$. So, $x = \dfrac{\sqrt{2}+x}{\sqrt{2}x}$ which yields to $\sqrt{2}x^2 - x - \sqrt{2} = 0$.
Solving for x using the quadratic formula yields to,

$$x = \frac{1 \pm \sqrt{1 - 4(\sqrt{2})(-\sqrt{2})}}{2\sqrt{2}} = \frac{1 \pm 3}{2\sqrt{2}}$$

Hence, if $f(0) = \sqrt{2}$ then either $f(2011) = \sqrt{2}$ or $f(2011) = -\dfrac{\sqrt{2}}{2}$.
However, suppose $f(2011) = -\dfrac{\sqrt{2}}{2}$. Consider $f(0) = f(2011 + (-2011)) = \dfrac{1}{f(2011)} + \dfrac{1}{f(-2011)}$ which implies that $\sqrt{2} = -\sqrt{2} + \dfrac{1}{f(-2011)}$. Thus, $f(-2011) = \dfrac{\sqrt{2}}{4}$.
But if we consider $f(-2011) = f(-2011+0) = \dfrac{1}{f(-2011)} + \dfrac{1}{f(0)}$, this means that $\dfrac{\sqrt{2}}{4} = 2\sqrt{2} + \dfrac{1}{f(0)}$.
Thus, $f(0) = -\dfrac{2\sqrt{2}}{7}$ which is a contradiction. Thus for $f(0) = \sqrt{2}$, $f(2011) = \sqrt{2}$.
If $f(0) = -\sqrt{2}$ then $x = f(2011) = f(2011+0) = \dfrac{1}{f(2011)} + \dfrac{1}{f(0)} = \dfrac{1}{x} - \dfrac{1}{\sqrt{2}}$.
So, $x = \dfrac{\sqrt{2}-x}{\sqrt{2}x}$ which yields to $\sqrt{2}x^2 + x - \sqrt{2} = 0$.
Solving for x using the quadratic formula yields to,

$$x = \frac{-1 \pm \sqrt{1 - 4(\sqrt{2})(-\sqrt{2})}}{2\sqrt{2}} = \frac{-1 \pm 3}{2\sqrt{2}}$$

Hence, if $f(0) = -\sqrt{2}$ then either $f(2011) = -\sqrt{2}$ or $f(2011) = \dfrac{\sqrt{2}}{2}$.
However, suppose $f(2011) = \dfrac{\sqrt{2}}{2}$. Consider $f(0) = f(2011 + (-2011)) = \dfrac{1}{f(2011)} + \dfrac{1}{f(-2011)}$ which implies that $-\sqrt{2} = \sqrt{2} + \dfrac{1}{f(-2011)}$. Thus, $f(-2011) = -\dfrac{\sqrt{2}}{4}$.
But if we consider $f(-2011) = f(-2011+0) = \dfrac{1}{f(-2011)} + \dfrac{1}{f(0)}$, this means that $-\dfrac{\sqrt{2}}{4} = -2\sqrt{2} + \dfrac{1}{f(0)}$.
Thus, $f(0) = \dfrac{2\sqrt{2}}{7}$ which is a contradiction. Thus for $f(0) = -\sqrt{2}$, $f(2011) = -\sqrt{2}$.
So the possible values for $f(2011)$ are $\boxed{\pm\sqrt{2}}$.