PHILIPPINE MATHEMATICAL OLYMPIAD 2015

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24 January 2015

University of Santo Tomas, Manila

SCHEDULE

7:30 AM – 8:30 AM	Registration
9:00 AM – 12:00 NN	Phase I : Written Phase
12:00 NN – 1:00 PM	Lunch Break
1:00 PM – 5:00 PM	Phase II : Oral Phase
	National Anthem
	Welcoming Remarks
	Oral Competition
6:30 PM – 8:30 PM	Dinner and Awarding Ceremonies

ABOUT THE PMO

First held in **1984**, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: **(1)** to awaken greater interest in and promote the appreciation of mathematics among students and teachers; **(2)** to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; **(3)** to provide a vehicle for the professional growth of teachers; and **(4)** to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the **International Mathematical Olympiad (IMO).** It is followed by the **Mathematical Olympiad Summer Camp (MOSC)**, a five-phase program for the twenty national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the seventeenth since 1984. Three thousand two hundred sixteen (3216) high school students from all over the country took the qualifying examination, out of these, two hundred thirteen (213) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty and these twenty students will compete for the top three positions and hopefully move on to represent the country in the **56th IMO**, which will be held in **Chiang Mai, Thailand**, from **4 to 16 July 2015** .

MESSAGE

According to Oswald Veblen, mathematics is "one of the essential emanations of the human spirit; a thing to be valued in and for itself, like art or poetry."

Math is as natural as breathing and sleeping; while its theories are formulaic and very complex, it appeals to the simplest nature of humans to measure and place values on both material and metaphysical things.

The Mathematical Society of the Philippines, through the 17th Philippine Mathematical Olympiad, continues to emphasize the importance of mathematics as more than just an academic field that breeds the next generation of engineers, architects, accountants or IT professionals, but an area of study that exposes our own mortality and promotes a better understanding of the physical world we live in.

As we prepare lifelong learners who are competitive in technical and industrial fields, we also hone critical thinkers who are immersed and proactive in addressing issues in the society. The Department of Education is looking forward to more partnerships with MSP to achieve this goal.

Tayo para sa edukasyon at kalinangan ng bawat batang Pilipino!

BR. ARMIN A. LUISTRO, FSC

Secretary

Department of Education

MESSAGE

Greetings of peace!

With the ushering of a new year, we welcome the much-awaited start of the Final Stage of the 17th Philippine Mathematical Olympiad (PMO), the country's oldest and hardest mathematics competition.

The PMO has continuously proven its great value in realizing and advancing exemplary talent in mathematics. It has honed the country's contingents to perform at a high level in the International Mathematics Olympiad (IMO), the most prestigious mathematics competition in the world. For seven straight years, our students have brought home two silvers and l3 bronzes, along with seven honourable mentions. This is a clear indication that the elusive gold is not far for the Philippines. The high degree of competition in the PMO brings out the best in our students in mathematics.

The excellence in mathematics shown by our young students drives the Science Education Institute (SEI) to continue supporting competitions like the PMO in honing our students in the exciting field of mathematics. SEI shall continue upholding its commitment in supporting the Mathematics Society of the Philippines in helping nurture the future mathematicians of the country.

We are optimistic that this year's PMO will harvest a new crop of outstanding change-makers in the S&T landscape. We hope that our participants will ultimately dedicate their gifts in service of the people through science and mathematics.

JOSETTE T. BIYO

Director

Science Education Institute, DOST

MESSAGE

Greetings to all participants in the National Finals of the 17th Philippine Mathematical Olympiad! The Mathematical Society of the Philippines is proud to once again implement this activity. The PMO, the country's most prestigious national mathematical competition seeks to identify our country's most talented high school students. I hope that the National Finals will be a challenging, memorable and fun experience for everyone. From your group the members of the Philippine team to the International Mathematical Olympiad will again be selected.

In recent years, our students may have felt a mounting pressure to achieve highly and demonstrate their very best. This is likely due to the continuous string of outstanding performances by our Philippine teams to the IMO in the last few years. But I am confident that our finalists have been guided and nurtured well by their coaches, parents and schools. I thank the latter for your constant care and support to your children and students.

To the students aspiring to be this year's PMO champion, remember what Benoit Mandelbrot, the Father of Fractal Theory once said – "*Science would be ruined if (like sports) it were to put competition above everything else.*" Instead, enjoy the competition, learn from it, and bring home positive memories from your experience.

Mabuhay kayong lahat!

JOSE MARIA P. BALMACEDA, PH. D.

President

Mathematical Society of the Philippines

MESSAGE

It is a privilege for our company, Collins International Trading Corporation, distributor of Sharp calculators in the Philippines, to be part of this prestigious undertaking. It is our advocacy to enhance math and science education by infusing technology.

We have witnessed the eagerness of many young mathematicians to be chosen as one of the Top 20 P.M.O. finalists. It is indeed a tedious process for these students. As we all know, aside from their talents, perseverance and determination motivated them to gear towards excellence.

Parents like me are so grateful to organizations that continue to develop the mathematical skills of our youth. Congratulations to the officers of the Math Society of the Philippines.

The greater the sacrifice, the greater the reward.

"Your work shall be rewarded, says the LORD." - Jeremiah 31:16

LUCERO ONG

Assistant Vice President
Sharp Calculators
Collins International Trading Corp.

MESSAGE

On behalf of FUSE's Board of Trustees, I would like to congratulate the top 20 performers in the Area Stage, including their coaches and teachers, who will compete in the National Stage of the 17th Philippine Mathematical Olympiad (PMO).

My warmest greetings, too, to the organizers, the Mathematical Society of the Philippines (MSP) and the Science Education Institute of the Department of Science and Technology for tirelessly promoting and upgrading math and science education through the holding of this prestigious annual event, which is participated in by public and private high school students in the country.

We, at FUSE, consider it a privilege and an honor to be a part of this laudable undertaking, the country's oldest and most challenging math competition. The Board hopes that the MSP and PMO would continue to arouse greater interest in mathematics among students and teachers, and our involvement will be replicated by others in the private sector, in order to assist MSP and PMO realize their goals.

The best of luck to the 20 competitors and to those who would qualify for the next math Olympiad: May you surpass the record-breaking achievements of the Philippine team to the international event held in South Africa last July. Through hard work, diligent training and preparation, I strongly believe you can bring more medals, possibly the elusive gold, and honor, as well, to our country.

Your impressive showing or victory, as underscored by our overachieving national team in the 55th math Olympiad, will make other countries take further notice that the Philippines' math and science education is on the right track and getting better.

May our continuing partnership spell more success as we pursue our shared goal to upgrade the standard of education in these two fields of discipline and I look forward to more fruitful endeavours with MSP and PMO.

Congratulations.

DR. LUCIO C. TAN

Vice-Chairman, Board of Trustees

Foundation of Upgrading the Standard of Education Inc.

MESSAGE

Congratulations to the Mathematical Society of the Philippines for holding the 17th Philippine Mathematical Olympiad. We commend you for having gone this far in helping Filipino students become globally competitive in the field of mathematics.

We from C & E Publishing, Inc., a leading provider of quality educational resources for the academic and professional sectors, are very proud to be part of your advocacy towards awakening greater interest in, and promoting the appreciation of, mathematics among students and teachers.

For five years now, we have been privileged to be one with your organization in providing Filipino students with the inspiration and motivation to develop not only their critical thinking and problem-solving skills, but also the core values that will help them succeed as future leaders of our country.

We wish the students, teachers, and organizers of the Philippine Mathematical Olympiad continued success in the national competition, and most especially, in the international tournament where we hope to earn the merit for the Philippines as the home of the best and brightest young mathematicians.

More power to your organization!

JOHN EMYL EUGENIO

Chief Operating Officer

C & E Publishing, Inc.

The PMO Team

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Alexis Jerome Lauan
Ma. Crisnalli Umali
Richard See

The PMO Team

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NCR
Lemuel Martin

AREA STAGE WINNERS

LUZON

1	Albert John Patupat	Holy Rosary College
2	Vince Jan Torres	Santa Rosa Science and Technology High School
3	Sephi Marz Liclican	Philippine Science High School - Ilocos Region

VISAYAS

1	Myles Denzel Delatore	Bethany Christian School
2	Elizalde Miguel Flores	Philippine Science High School - Western Visayas
2	Andrew Thomas Yu	St. John's Institute
3	Jua Park	Sacred Heart School - Ateneo de Cebu

MINDANAO

1	Xavier Jefferson Ray Go	Zamboanga Chong Hua High School
2	Gene Go Jr.	Zamboanga Chong Hua High School
3	Sean Anderson Ty	Zamboanga Chong Hua High School

NCR

1	Kyle Patrick Dulay	Philippine Science High School - Main
2	Adrian Reginald Sy	Saint Jude Catholic School
3	Farell Eldrian Wu	MGC New Life Christian Academy

~ PRIZES ~

The prizes for the TOP THREE for each Area/Region (Luzon, Visayas, Mindanao, NCR) are:

FIRST PLACE - Medal and SHARP EL-W531XH Calculator

SECOND PLACE - Medal and SHARP EL-510RN Calculator

THIRD PLACE - Medal and SHARP EL-501X Calculator

The prizes for the TOP THREE in the NATIONAL FINAL STAGE are:

FIRST PLACE - P 20,000, Medal, and SHARP EL-9950 Calculator with SHARP Goodies

SECOND PLACE - P 15,000, Medal, and SHARP EL-W506X Calculator with SHARP Goodies

THIRD PLACE - P 10,000, Medal, and SHARP EL-W506X Calculator with SHARP Goodies

The coaches of the first, second and third placers will receive P 5,000, P 3,000, and P 2,000, respectively and SHARP Goodies.

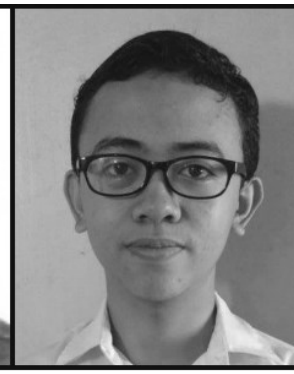
The PMO Finalists

CLYDE WESLEY ANG
Chiang Kai Shek College

LESLEY CLARICE BALETE
Grace Christian College

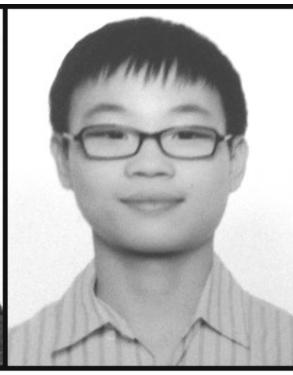
KYLE PATRICK DULAY
Philippine Science High School – Main

RAYMOND JOSEPH FADRI
Makati Science High School



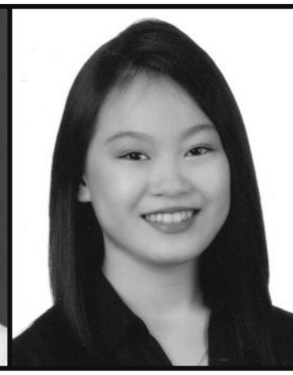
CHRISTIAN PHILIP GELERA
Philippine Science High School – Main

ANDREA JESSICA JABA
Saint Jude Catholic School

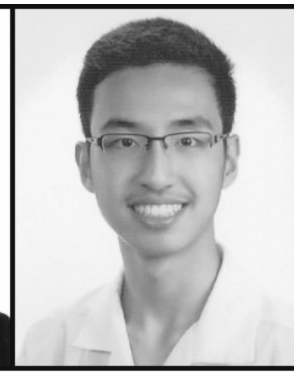


SEDRICK SCOTT KEH
Xavier School

SEPHI MARZ LICLICAN
Philippine Science High School – Ilocos Region



KELSEY LIM TIONG SOON
Grace Christian College

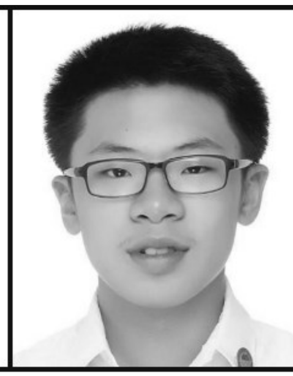


ANDREW BRANDON ONG
Chiang Kai Shek College

GERALD PASCUA
Philippine Science High School – Main

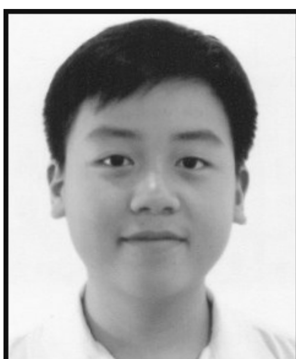
ALBERT JOHN PATUPAT
Holy Rosary College

SHAQUILLE WYAN QUE
Grace Christian College



ADRIAN REGINALD SY
Saint Jude Catholic School

ANDREW LAWRENCE SY
Xavier School



HANS MARKSON TAN
St. Stephen's High School

ISABELLA MAE TAN
Saint Jude Catholic School

GEN MARK TANNO
Southville International School & Colleges

VINCE JAN TORRES
Santa Rosa Science and Technology High School

FARELL ELDRIAN WU
MGC New Life Christian Academy

PMO Highlights

QUALIFYING STAGE

11 OCTOBER 2014

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $f(x) = 1 + \frac{1}{x}$, find $f(f(f(1)))$.

   (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) $\frac{5}{4}$

2. What is the units digit of the product of five consecutive integers?

   (a) 0 (b) 2 (c) 4 (d) 6

3. In an urn, 4/7 of the chips are red and the rest are blue. If the number of red chips is reduced by half and the number of blue chips is doubled, what is now the fraction of red chips in the urn?

   (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{2}{7}$

4. If $p = 1 + 3 + \cdots + 2011 + 2013$ and $q = 2 + 4 + \cdots + 2012 + 2014$, evaluate $q - p$.

   (a) -1008 (b) 0 (c) 1007 (d) 1008

5. Let the sum of N numbers be N. If N is added to each of these numbers, and then each is multiplied by N, what is now the sum of these N resulting numbers?

   (a) $2N^2$ (b) $N^2 + N^3$ (c) $2N^3$ (d) N^4

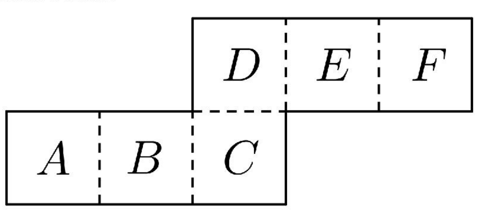
6. If $r + s - a - b = 2$ and $rs + a + b + 2 = 0$, find the value of $(r + 1)(s + 1)$.

   (a) -1 (b) 0 (c) 2 (d) 1

7. A group of 10 women and 7 men are to be arranged at random in a row. What is the probability that all of the women will be beside each other?

   (a) $\frac{1}{\binom{17}{10}}$ (b) $\frac{10!}{\binom{17}{10}}$ (c) $\frac{10!7!}{17!}$ (d) $\frac{10!8!}{17!}$

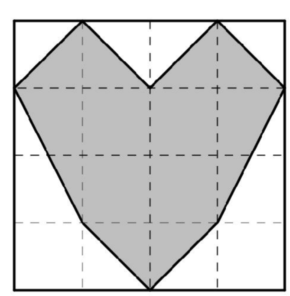
8. When the figure below is folded along the dotted segments to form a cube, which pair of letters will be opposite each other?



   (a) (A, C) (c) (C, E)
   (b) (B, F) (d) (C, F)

9. In the square on the right, the dotted segments divide each side into four equal parts. If the square has area 256 square units, what is the area of the shaded octagon, in square units?

   (a) 144 (b) 160 (c) 176 (d) 192



10. A huge urn contains as many marbles as needed of the following three colors: blue, red and yellow. Each blue marble weighs 42 grams, each red marble weighs 28 grams, and each yellow marble weighs 20 grams. Marbles of each color are to be distributed in three bags such that each bag contains marbles of a single color, and all bags have the same weight. What is the least number of marbles needed to do this?

    (a) 36 (b) 42 (c) 46 (d) 52

11. How many zeros does $f(x) = \log(\sin x)$ have in the interval $[0, 4\pi]$?

    (a) 3 (b) 2 (c) 1 (d) none

12. Suppose that x is a number such that $\frac{9}{a} + a \geq x$ for any positive number a. What is the largest possible value of x?

    (a) 3 (b) 4 (c) 6 (d) 9

13. In the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2014$, x is used infinitely many times as exponent. Which of the following is a root of the equation?

    (a) $\log_{2014} 2014$ (b) $\sqrt{2014}$ (c) $\sqrt[2014]{2014}$ (d) 2014^{2014}

14. A point P is outside a circle and on the same plane as it. If the points on the circle closest and farthest from P are 4 and 16 units away, respectively, how long is a tangent segment from P to the circle?

    (a) 6 (b) 8 (c) 10 (d) 12

15. How many real solutions does the equation have?

$$\log_{2014} x - \log_x 2014 = \log_{\sqrt[2014]{2014}} \sqrt[2014]{x}$$

    (a) none (b) 1 (c) 2 (d) infinitely many

PART II. Choose the best answer. Each correct answer is worth three points.

1. The interior angles of a convex polygon are all 160°, except for one which is $x°$. What is the smallest possible value of x, if the polygon has an even number of sides?

    (a) 10 (b) 20 (c) 40 (d) 80

2. For the function f, $f(2x) = x^2 + x - 2$ for all real numbers x. Let a and b be the sum and product, respectively, of the roots of the equation $f(x/2) = 4$. Find $a + b$.

    (a) 96 (b) -96 (c) 100 (d) -100

3. Find the length of the shortest path on the plane from $P(0, 0)$ to $Q(2, 1)$, so that any point on this path is at least one unit away from $(1, 0)$, $(1, 1)$, $(1, 2)$ and $(2, 0)$.

    (a) 2π (b) $1 + \frac{3}{2}\pi$ (c) $1 + \sqrt{2} + \pi$ (d) $\frac{11}{6}\pi$

4. Simplify:

$$\frac{1+2+3}{1+2+3+4} \times \frac{1+2+3+4+5}{1+2+3+4+5+6} \times \cdots \times \frac{1+2+\cdots+19}{1+2+\cdots+19+20}.$$

(a) $\frac{1}{5}$ (b) $\frac{3}{20}$ (c) $\frac{1}{7}$ (d) $\frac{1}{35}$

5. An infinite geometric series has sum 2014. If the sum of their squares is also 2014, find the first term.

   (a) $\frac{2013}{2014}$ (b) $\frac{2013}{2015}$ (c) $\frac{2014}{2015^2}$ (d) $\frac{4028}{2015}$

6. Let the longest diagonal of a closed rectangular box be 6 units in length. If the lengths of its sides are all integers, find the surface area of the box in square units.

   (a) 48 (b) 64 (c) 76 (d) 96

7. For how many integers $n > 0$ will the sum of the first n positive integers be a factor of $8n^2$?

   (a) 4 (b) 5 (c) 6 (d) infinitely many

8. For which value of the constant k below will the inequality

$$9k^2(x-5)^2 - 125k^2 \geq (9+5k^2)(x^2-10x)+225$$

   have a unique solution?

   (a) $\frac{1}{2014}$ (b) $\frac{3}{2}$ (c) -9 (d) 2014

9. How many values of the integer k will make the triangle with sides 6, 8 and k obtuse?

   (a) 3 (b) 6 (c) 8 (d) 11

10. How many polynomials

$$x^{2014} + a_{2013}x^{2013} + a_{2012}x^{2012} + \ldots + a_2x^2 + a_1x + a_0$$

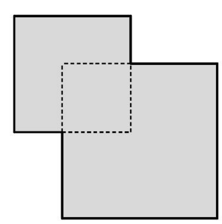
   with real coefficients $a_0, a_1, \ldots, a_{2013}$ can be formed, if all its zeros are real and can only come from the set $\{1, 2, 3\}$?

   (a) 2014^3 (b) $\binom{2014}{2}$ (c) $\binom{2015}{3}$ (d) $\binom{2016}{2}$

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Solve the equation

$$(2 \cdot 3^x)^3 + (9^x - 3)^3 = (9^x + 2 \cdot 3^x - 3)^3.$$



2. Two squares have integer side lengths which are in the ratio 4 : 3. Their intersection is also a square with integer side length. If the total area shown on the surface is equal to 5000 square units, how long is a side of the largest square?

3. How many 3-element subsets of $\{1, 2, 3, \ldots, 11, 12, 13\}$ are there for which the sum of the 3 elements is divisible by 3?

4. A sequence $a_1, a_2, a_3, \ldots$ is defined in the following manner: $a_1 = 1$, and for every integer $n \geq 2$, a_n is the nth even integer greater than a_{n-1}. Find the remainder when a_{2014} is divided by 2014.

5. Let α, β and γ be the roots of $x^3 - 4x - 8 = 0$. Find the numerical value of the expression

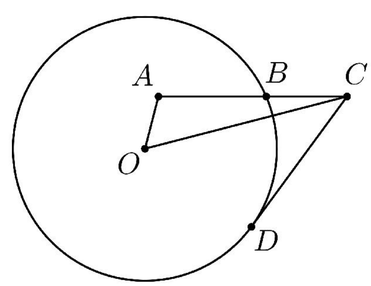
$$\frac{\alpha+2}{\alpha-2} + \frac{\beta+2}{\beta-2} + \frac{\gamma+2}{\gamma-2}.$$

AREA STAGE

15 NOVEMBER 2014

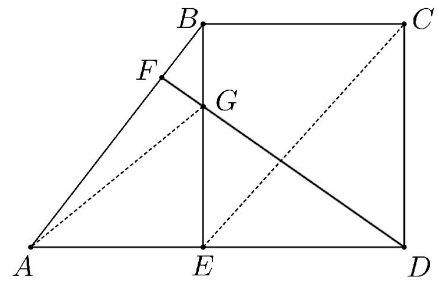
PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. What is the fourth smallest positive integer having exactly 4 positive integer divisors, including 1 and itself?
2. Let $f(x) = a^x - 1$. Find the largest value of $a > 1$ so that if $0 \leq x \leq 3$, then $0 \leq f(x) \leq 3$.
3. Simplify the expression $\left(1 + \frac{1}{i} + \frac{1}{i^2} + \ldots + \frac{1}{i^{2014}}\right)^2$.
4. Find the numerical value of $(1 - \cot 37°)(1 - \cot 8°)$.
5. Triangle ABC has a right angle at B, with $AB = 3$ and $BC = 4$. If D and $E$are points on AC and BC, respectively, such that $CD = DE = \frac{5}{3}$, find the perimeter of quadrilateral $ABED$.
6. Rationalize the denominator of $\dfrac{6}{\sqrt[3]{4} + \sqrt[3]{16} + \sqrt[3]{64}}$ and simplify.
7. Find the area of the triangle having vertices $A(10, -9)$, $B(19, 3)$, and $C(25, -21)$.
8. How many ways can 6 boys and 6 girls be seated in a circle so that no two boys sit next to each other?
9. Two numbers p and q are both chosen randomly (and independently of each other) from the interval $[-2, 2]$. Find the probability that $4x^2 + 4px + 1 - q^2 = 0$ has imaginary roots.
10. In $\triangle ABC$, $\angle A = 80°$, $\angle B = 30°$, and $\angle C = 70°$. Let BH be an altitude of the triangle. Extend BH to a point D on the other side of AC so that $BD = BC$. Find $\angle BDA$.
11. Find all integer values of n that will make $\dfrac{6n^3 - n^2 + 2n + 32}{3n + 1}$ an integer.
12. Suppose that the function $y = f(x)$ satisfies $1 - y = \dfrac{9e^x + 2}{12e^x + 3}$. If m and n are consecutive integers so that $m < \dfrac{1}{y} < n$ for all real x, find the value of mn.
13. The product of the two roots of $\sqrt{2014}x^{\log_{2014} x} = x^{2014}$ is an integer. Find its units digit.
14. In how many ways can Alex, Billy, and Charles split 7 identical marbles among themselves so that no two have the same number of marbles? It is possible for someone not to get any marbles.
15. In a Word Finding game, a player tries to find a word in a 12×12 array of letters by looking at blocks of adjacent letters that are arranged horizontally, arranged vertically, or arranged diagonally. How many such 3-letter blocks are there in a given 12×12 array of letters?
16. Find the largest possible value of
$$(\sin \theta_1)(\cos \theta_2) + (\sin \theta_2)(\cos \theta_3) + \cdots + (\sin \theta_{2013})(\cos \theta_{2014}) + (\sin \theta_{2014})(\cos \theta_1).$$
17. What is the remainder when
$$16^{15} - 8^{15} - 4^{15} - 2^{15} - 1^{15}$$
is divided by 96?



18. (*See figure on the right.*) Segment CD is tangent to the circle with center O, at D. Point A is in the interior of the circle, and segment AC intersects the circle at B. If $OA = 2$, $AB = 4$, $BC = 3$, and $CD = 6$, find the length of segment OC.
19. Find the maximum value of
$$(1 - x)(2 - y)(3 - z)\left(x + \frac{y}{2} + \frac{z}{3}\right)$$
where $x < 1$, $y < 2$, $z < 3$, and $x + \frac{y}{2} + \frac{z}{3} > 0$.

20. Trapezoid $ABCD$ has right angles at C and D, and $AD > BC$. Let E and F be the points on AD and AB, respectively, such that $\angle BED$ and $\angle DFA$ are right angles. Let G be the point of intersection of segments BE and DF. If $\angle CED = 58°$ and $\angle FDE = 41°$, what is $\angle GAB$?



PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Arrange these four numbers from smallest to largest: $\log_3 2$, $\log_5 3$, $\log_{625} 75$, $\frac{2}{3}$.

2. What is the greatest common factor of all integers of the form $p^4 - 1$, where p is a prime number greater than 5?

3. Points A, M, N and B are collinear, in that order, and $AM = 4$, $MN = 2$, $NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.

ANSWER KEY

QUALIFYING STAGE

PART I.

1. C
2. A
3. C
4. C
5. B
6. D
7. D
8. A
9. A
10. C
11. B
12. C
13. C
14. B
15. A

PART II.

1. C
2. D
3. B
4. C
5. D
6. B
7. A
8. A
9. B
10. D

AREA STAGE

PART I.

1. 14
2. $\sqrt[3]{4}$
3. -1
4. 2
5. $\dfrac{28}{3} = 9\dfrac{1}{3}$
6. $2 - \sqrt[3]{2}$
7. 144
8. $6!5! = 86400$
9. $\dfrac{\pi}{16}$
10. 70°
11. 0, 10
12. 12
13. 6
14. 24
15. 440
16. 1007
17. 31
18. $\sqrt{60} = 2\sqrt{15}$
19. $\dfrac{3^5}{2^7} = \dfrac{243}{128}$
20. 17°

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Arrange these four numbers from smallest to largest: $\log_3 2$, $\log_5 3$, $\log_{625} 75$, $\frac{2}{3}$.

   Solution. The numbers, arranged from smallest to largest, are $\log_3 2$, $\frac{2}{3}$, $\log_{625} 75$, and $\log_5 3$.

   - Since $\left(3^{\log_3 2}\right)^3 = 8$ and $\left(3^{\frac{2}{3}}\right)^3 = 9$, then $\log_3 2 < \frac{2}{3}$.
   - Since $\left(625^{\frac{2}{3}}\right)^3 = 5^8 = 5^6 \cdot 25$ and $\left(625^{\log_{625} 75}\right)^3 = 75^3 = 5^6 \cdot 27$, then $\frac{2}{3} < \log_{625} 75$.
   - If $A = \log_{625} 75$, then $5^{4A} = 75$. On the other hand, $5^{4\log_5 3} = 81$. Thus, $\log_{625} 75 < \log_5 3$.

2. What is the greatest common factor of all integers of the form $p^4 - 1$, where p is a prime number greater than 5?

   Solution. Let $f(p) = p^4 - 1 = (p-1)(p+1)(p^2+1)$. Note that $f(7) = 2^5 \cdot 3 \cdot 5^2$ and $f(11) = 2^4 \cdot 3 \cdot 5 \cdot 61$. We now show that their greatest common factor, $2^4 \cdot 3 \cdot 5$, is actually the greatest common factor of all numbers $p^4 - 1$ so described.

   - Since p is odd, then $p^2 + 1$ is even. Both $p - 1$ and $p + 1$ are even, and since they are consecutive even integers, one is actually divisible by 4. Thus, $f(p)$ is always divisible by 2^4.
   - When divided by 3, p has remainder either 1 or 2.

- If $p \equiv 1$, then $3|p-1$.
- If $p \equiv 2$, then $3|p+1$.

Thus, $f(p)$ is always divisible by 3.

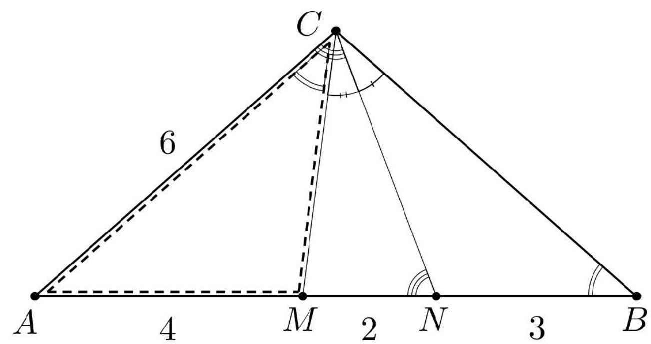
- When divided by 5, p has remainder 1, 2, 3 or 4.
  - If $p \equiv 1$, then $5|p-1$.
  - If $p \equiv 2$, then $p^2+1 \equiv 2^2+1 = 5 \equiv 0$.
  - If $p \equiv 3$, then $p^2+1 \equiv 3^2+1 = 10 \equiv 0$.
  - If $p \equiv 4$, then $5|p+1$.

  Thus, $f(p)$ is always divisible by 5.

Therefore, the greatest common factor is $2^4 \cdot 3 \cdot 5 = 240$.

3. Points A, M, N and B are collinear, in that order, and $AM = 4$, $MN = 2$, $NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.

Solution.



Since $\dfrac{CA}{AM} = \dfrac{3}{2} = \dfrac{BA}{AC}$ and $\angle CAM = \angle BAC$, then $\triangle CAM \sim \triangle BAC$. Therefore,

$$\angle MCA = \angle CBA. \tag{1}$$

Since $AC = 6 = AN$, then $\triangle CAN$ is isosceles. Therefore,

$$\angle ACN = \angle ANC. \tag{2}$$

Thus,

$$\begin{aligned} \angle BCN &= \angle ANC - \angle CBA && \text{since } \angle ANC \text{ is an exterior angle of } \triangle BNC \\ &= \angle ACN - \angle MCA && \text{using (1) and (2)} \\ &= \angle MCN. \end{aligned}$$

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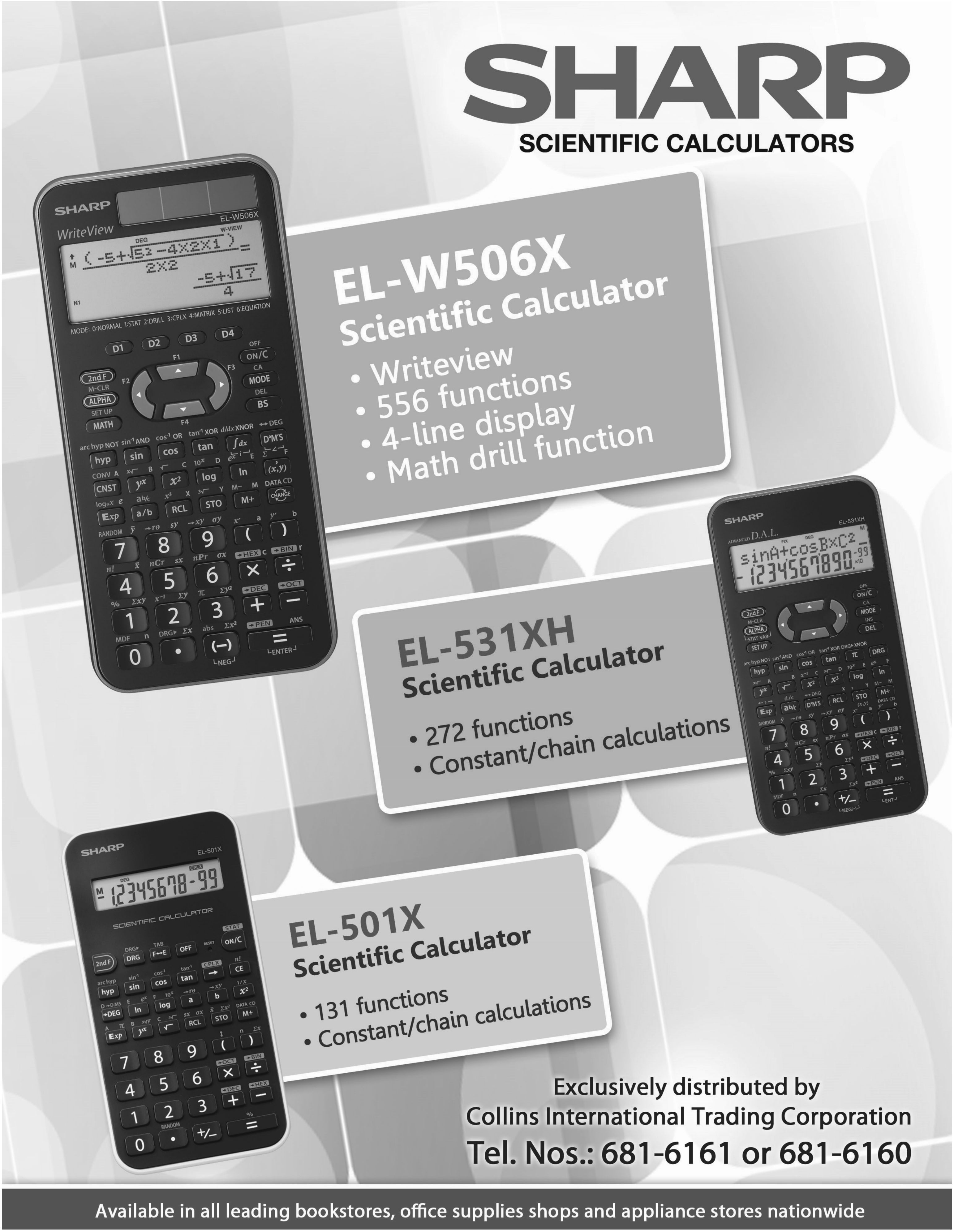
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