12^{th} Philippine Mathematical Olympiad

National Stage, 23 January 2010

Oral Phase

15.1. What is the smallest positive integral value of n such that $n^{300} > 3^{500}$?

15.2. A figure consists of two overlapping circles that have radii 4 and 6. If the common region of the circles has area 2π, what is the area of the entire figure?

15.3. Find all real values of x that satisfy the equation $x^{x^{2010}} = x^{2010}$.

15.4. Both roots of the quadratic equation $x^2 - 30x + 13k = 0$ are prime numbers. What is the largest possible value of k?

15.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies the functional equation

$$f(x - y) = 2009 f(x) f(y)$$

for all $x, y \in \mathbb{R}$. If $f(x)$ is never zero, what is $f(\sqrt{2009})$?

15.6. If the parabola $y + 1 = x^2$ is rotated clockwise by $90°$ about its focus, what will be the new coordinates of its vertex?

15.7. How many ways can you choose four integers from the set $\{1, 2, 3, \ldots, 10\}$ so that no two of them are consecutive?

15.8. Let ABC be a triangle with $AB = 12$, $BC = 16$, and $AC = 20$. Compute the area of the circle that passes through C and the midpoints of AB and BC.

15.9. Which real numbers x satisfy the inequality $|x - 3| \geq |x|$?

15.10. Let $\log_{14} 16$ be equal to a. Express $\log_8 14$ in terms of a.

15.11. Find the values of a and b such that $ax^4 + bx^2 + 1$ is divisible by $x^2 - x - 2$.

15.12. What is the probability that a randomly chosen positive divisor of 2010 has two digits?

15.13. Let $[\![x]\!]$ denote the greatest integer less than or equal to the real number x. What is the largest two-digit integral value of $x[\![x]\!]$?

15.14. How many times does the graph of $y + 1 = \left|\log_{1/2} |x|\right|$ cross the x-axis?

15.15. Considered to be the most prolific mathematician of all time, he published, in totality, the most number of mathematical pages in history. Undertaken by the Swiss Society of Natural Sciences, the project of publishing his collected works is still going on and will require more than 75 volumes. Who is this great mathematician Switzerland has produced?

30.1. The nonzero numbers x, y, and z satisfy the equations

$$xy = 2(x+y), \quad yz = 4(y+z), \quad \text{and} \quad xz = 8(x+z).$$

Solve for x.

30.2. The positive integers are grouped as follows:

$$A_1 = \{1\},\ A_2 = \{2,3,4\},\ A_3 = \{5,6,7,8,9\},\ \text{and so on.}$$

In which group does 2009 belong to?

30.3. Triangle ABC is right-angled at C, and point D on AC is the foot of the bisector of $\angle B$. If $AB = 6$ cm and the area of $\triangle ABD$ is 4.5 cm^2, what is the length, in cm, of CD?

30.4. For each positive integer n, let S_n be the sum of the infinite geometric series whose first term is n and whose common ratio is $\frac{1}{n+1}$. Determine the least value of n such that

$$S_1 + S_2 + \cdots + S_n > 5150.$$

30.5. Let x and y be positive real numbers such that $x + 2y = 8$. Determine the minimum value of

$$x + y + \frac{3}{x} + \frac{9}{2y}.$$

30.6. Let d and n be integers such that $9n + 2$ and $5n + 4$ are both divisible by d. What is the largest possible value of d?

30.7. Find all integers n such that $5n - 7$, $6n + 1$, and $20 - 3n$ are all prime numbers.

30.8. When

$$(x^2 + 2x + 2)^{2009} + (x^2 - 3x - 3)^{2009}$$

is expanded, what is the sum of the coefficients of the terms with odd exponents of x?

30.9. If $0 < \theta < \pi/2$ and $1 + \sin\theta = 2\cos\theta$, determine the numerical value of $\sin\theta$.

30.10. For what real values of k does the system of equations

$$\begin{cases} x - ky = 0 \\ x^2 + y = -1 \end{cases}$$

have real solutions?

60.1. In $\triangle ABC$ with $BC = 24$, one of the trisectors of $\angle A$ is a median, while the other trisector is an altitude. What is the area of $\triangle ABC$?

60.2. How many integral solutions does the equation

$$|x| + |y| + |z| = 9$$

60.3. Let X, Y, and Z be points on the sides BC, AC, and AB of $\triangle ABC$, respectively, such that AX, BY, and CZ are concurred at point O. The area of $\triangle BOC$ is a. If $BX : XC = 2 : 3$ and $CY : YA = 1 : 2$, what is the area of $\triangle AOC$?

60.4. Find the only value of x in the open interval $(-\pi/2, 0)$ that satisfies the equation

$$\frac{\sqrt{3}}{\sin x} + \frac{1}{\cos x} = 4.$$

60.5. The incircle of a triangle has radius 4, and the segments into which one side is divided by the point of contact with the incircle are of lengths 6 and 8. What is the perimeter of the triangle?

Written Phase

1. Find all primes that can be written both as a sum of two primes and as a difference of two primes.

2. On a cyclic quadrilateral $ABCD$, there is a point P on side AD such that the triangle CDP and the quadrilateral $ABCP$ have equal perimeters and equal areas. Prove that two sides of $ABCD$ have equal lengths.

3. Let $\mathbb{R}^\star$ be the set of all real numbers, except 1. Find all functions $f : \mathbb{R}^\star \to \mathbb{R}$ that satisfy the functional equation

$$x + f(x) + 2f\left(\frac{x + 2009}{x - 1}\right) = 2010.$$

4. There are 2008 blue, 2009 red, and 2010 yellow chips on a table. At each step, one chooses two chips of different colors, and recolor both of them using the third color. Can all the chips be of the same color after some steps? Prove your answer.

5. Determine, with proof, the smallest positive integer n with the following property: For every choice of n integers, there exist at least two whose sum or difference is divisible by 2009.

Answers

Oral Phase

15.1. 7

15.2. 50π

15.3. $\sqrt[2010]{2010}$, $-\sqrt[2010]{2010}$, 1

15.4. 17

15.5. $\frac{1}{2009}$

15.6. $(-\frac{3}{4}, -\frac{1}{4})$

15.7. 35

15.8. 25π

15.9. $(-\infty, 3/2]$

15.10. $\frac{4}{3a}$

15.11. $a = 1/4$, $b = -5/4$

15.12. $\frac{1}{4}$

15.13. 99

15.14. 4

15.15. Leonhard Euler

30.1. $16/3$

30.2. A_{45}

30.3. 1.5

30.4. 101

30.5. 8

30.6. 26

30.7. only $n = 6$

30.8. -1

30.9. $3/5$

30.10. $-\frac{1}{2} \leq x \leq \frac{1}{2}$

60.1. $32\sqrt{3}$

60.2. 326

60.3. $3a$

60.4. $-4\pi/9$

60.5. 42

Oral Phase

1. Let p be a prime that can be written as a sum of two primes and as a difference of two primes. Clearly, we have $p > 2$. Then p must be odd, so that $p = q + 2 = r - 2$ for some odd primes q and r.

We consider three cases.

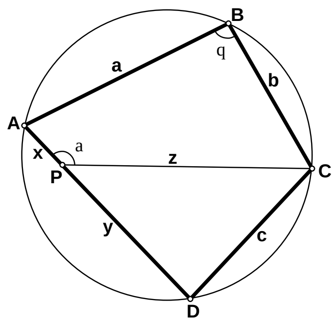
Case 1. Suppose that $q \equiv 1 \pmod 3$. Then p is a multiple of 3, implying that $p = 3$. It follows that $p = 3$, which means that $q = 1$, a contradiction.

CASE 2. Suppose that $q \equiv 2 \pmod 3$. Then $r \equiv 0 \pmod 3$, which implies that $r = 3$. This leads to $p = 1$, which is again a contradiction.

CASE 3. Suppose that $q \equiv 0 \pmod 3$. Then $q = 3$, and it follows that $p = 5$ and $r = 7$.

From the above three cases, $p = 5$ is the only prime that is a sum of two primes and a difference of two primes. □

2. We denote by (XYZ) and $(WXYZ)$ the areas of $\triangle XYZ$ and quadrilateral $WXYZ$, respectively. We use the labels depicted in the following figure.



With equal perimeters, we get

$$a + b + z + x = c + y + z$$

or

$$a + b + x = c + y. \tag{1$\star$}$$

With equal areas, we get

$$(ABC) + (ACP) = (CDP). \tag{2$\star$}$$

Since $\triangle ACP$ and $\triangle CDP$ have the same altitude from C, we have

$$\frac{(ACP)}{(CDP)} = \frac{x}{y} \qquad \Longrightarrow \qquad (ACP) = \frac{x}{y} \cdot (CDP).$$

With (2$\star$), we have

$$(ABC) = \left(1 - \frac{x}{y}\right)(CDP) = \frac{y - x}{y} \cdot (CDP). \tag{3$\star$}$$

On the other hand, since $ABCD$ is cyclic, we know that $\angle D = 180° - \theta$. Then $(ABC) = \frac{1}{2}ab\sin\theta$ and $(CDP) = \frac{1}{2}cy\sin(180° - \theta)$. After noting that $\sin(180° - \theta) = \sin\theta$ and applying (1⋆), equation (3⋆) reduces to

$$ab = c(a + b - c).$$

This last equation is equivalent to

$$(c - b)(c - a) = 0,$$

which implies that $b = c$ or $a = c$. □

3. Let $g(x) = \dfrac{x + 2009}{x - 1}$. Then the given functional equation becomes

$$x + f(x) + 2f(g(x)) = 2010. \qquad (1\star)$$

Replacing x with $g(x)$ in (1⋆), and after noting that $g(g(x)) = x$, we get

$$g(x) + f(g(x)) + 2f(x) = 2010. \qquad (2\star)$$

Eliminating $f(g(x))$ in (1⋆) and (2⋆), we obtain

$$x - 3f(x) - 2g(x) = -2010.$$

Solving for $f(x)$ and using $g(x) = \dfrac{x + 2009}{x - 1}$, we have

$$f(x) = \frac{x^2 + 2007x - 6028}{3(x - 1)}.$$

It is not difficult to verify that this function satisfies the given functional equation. □

4. After some steps, suppose that there a blue, b red, and c yellow chips on the table. We denote this scenario by the ordered triple (a, b, c). Then the next step produces $(a-1, b-1, c+2)$, $(a+2, b-1, c-1)$, or $(a-1, b+2, c-1)$. One crucial observation on these three possibilities is the fact that

$$(a-1)-(b-1) \equiv (a+2)-(b-1) \equiv (a-1)-(b+2) \equiv a-b \pmod 3;$$

that is, from one step to the next, the difference between the number of blue chips and the number of red chips does not change modulo 3.

Starting with $(2008, 2009, 2010)$, we verify if we can end up with $(6027, 0, 0)$, $(0, 6027, 0)$, or $(0, 0, 6027)$. Since $2008 - 2009 \equiv 2 \pmod 3$, but

$$6027 - 0 \equiv 0 - 6027 \equiv 0 - 0 \equiv 0 \pmod 3,$$

it follows that all the chips can never be of the same color after any number of steps. □

5. We show that the least integer with the desired property is 1006. We write $2009 = 2 \cdot 1004 + 1$.

Consider the set $\{1005, 1006, \ldots, 2009\}$, which contains 1005 integers. The sum of every pair of distinct numbers from this set lies between 2011 and 4017, none of which is divisible by 2009. On the other hand, the (absolute) difference between two distinct integers from this set lies between 1 and 1004, none of which again is divisible by 2009. It follows that the smallest integer with the desired property is at least 1006.

Let A be a set of 1006 integers. If there are two numbers in A that have the same remainder when divided by 2009, then we are done.

Suppose, on the contrary, that all the 1006 remainders of the integers in A modulo 2009 are all different. Thus, the set of remainders is a 1006-element subset of the set $\{0, 1, \ldots, 2008\}$. One can also consider the remainders as forming a 1006-element subset of the set $X = \{-1004, -1003, \ldots, -1, 0, 1, 2, \ \ldots, 1004\}$. Every 1006-element subset of X contains two elements whose sum is zero. Thus, A contains two numbers whose sum is divisible by 2009. Since $|A| = 1006$, we deduce that 1006 is the least integer with the desired property. □