$\boxed{\text{EASY}}$ 15 seconds, 2 points

1. What is the smallest number that is greater than 2015 and divisible by both 6 and 35?

   Answer: 2100

   Solution: The desired number is divisible by 2, 3, 5 and 7. Hence, it is the least multiple of $2 \times 3 \times 5 \times 7 = 210$ that is greater than 2015. The desired number is 2100.

2. A function $f(x)$ satisfies
$$(2-x)f(x) - 2f(3-x) = -x^3 + 5x - 18$$
   for all real numbers x. Solve for $f(0)$.

   Answer: $f(0) = 7$

   Solution: Set $x = 0$ and $x = 3$ to obtain the system
$$\begin{cases} 2f(0) - 2f(3) & = -18, \\ -f(3) - 2f(0) & = -30. \end{cases}$$
   We can solve for $f(0)$ from the system and obtain $f(0) = 7$.

3. Let $f(x) = \ln x$. What are the values of x in the domain of $(f \circ f \circ f \circ f \circ f)(x)$?

   Answer: $x > e^{e^e}$

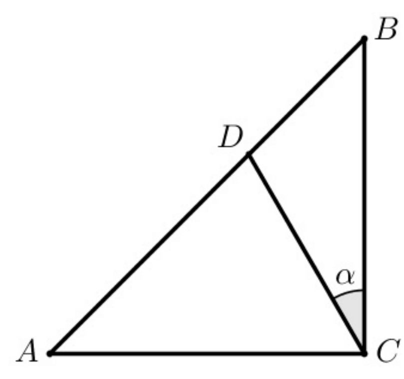
   Solution: We have $(f \circ f \circ f \circ f \circ f)(x) = \ln(\ln(\ln(\ln(\ln(x)))))$. The domain of this function must satisfy $\ln(\ln(\ln(\ln(x)))) > 0$ which implies that $\ln(\ln(\ln(x))) > 1$, $\ln(\ln(x)) > e$, $\ln(x) > e^e$ and $x > e^{e^e}$.

4. There are two distinct real numbers which are larger than their reciprocals by 2. Find the product of these numbers.

   Answer: -1

   Solution: Let x be one of these real numbers. We have $x = \frac{1}{x} + 2$, which is equivalent to $x^2 - 2x - 1 = 0$. The required real numbers are the two distinct roots of this quadratic equation, which have a product of -1.

5. In the figure on the right, $|AC| = |BC| = 1$ unit, $\alpha = 30°$, and $\angle ACB = 90°$. Find the area of $\triangle ADC$.



   Answer: $\frac{1}{4}(3 - \sqrt{3})$ (Please accept $\frac{1}{4}(12 - 6\sqrt{3})$, in case they used half-angle formulas instead.)

   Solution: Since $|AC| = |BC|$ and $\angle ACB = 90°$, it follows $\angle DAC = \angle CBD = 45°$. Because $\angle\alpha = 30°$ and $\angle ACD = 60°$, we have $\angle CDA = 75°$. Utilizing the Sine Law, we get
$$\frac{|CD|}{\sin 45°} = \frac{|AC|}{\sin 75°} = \frac{1}{\sin 75°}.$$
   This leads to
$$|CD| = \frac{\sin 45°}{\sin 75°} = \sqrt{3} - 1.$$

Then the altitude of $\triangle ADC$ with respect to the base AC has length

$$h = \left(\sqrt{3}-1\right) \times \sin 60^\circ = \frac{1}{2}(3-\sqrt{3}).$$

Therefore, the area of $\triangle ADC$ is $\frac{1}{4}(3-\sqrt{3})$ square units.

6. An urn contains five red chips numbered 1 to 5, five blue chips numbered 1 to 5, and five white chips numbered 1 to 5. Two chips are drawn from this urn without replacement. What is the probability that they have either the same color or the same number?

   Answer: $\frac{3}{7}$

   Solution: The required probability is

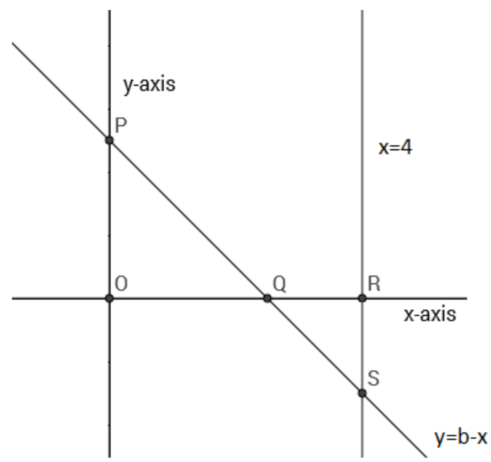
$$\frac{3 \cdot \binom{5}{2} + 5 \cdot \binom{3}{2}}{\binom{15}{2}} = \frac{30+15}{105} = \frac{3}{7}.$$

7. Let n be a positive integer greater than 1. If $2n$ is divided by 3, the remainder is 2. If $3n$ is divided by 4, the remainder is 3. If $4n$ is divided by 5, the remainder is 4. If $5n$ is divided by 6, the remainder is 5. What is the least possible value of n?

   Answer: 61

   Solution: We have $2n \equiv 2 \pmod 3$, $3n \equiv 3 \pmod 4$, $4n \equiv 4 \pmod 5$, and $5n \equiv 5 \pmod 6$. This is equivalent to saying that $n \equiv 1 \mod 3, 4, 5,$ and 6. The smallest such n is one more than the LCM of 3, 4, 5, and 6, which is 61.

8. In the figure on the right, the line $y = b - x$, where $0 < b < 4$, intersects the y-axis at P and the line $x = 4$ at S. If the ratio of the area of $\triangle QRS$ to the area of $\triangle QOP$ is $9 : 25$, determine the value of b.



   Answer: $b = \frac{5}{2}$

   Solution: It is easy to see that $\triangle QRS$ has legs $4-b$ and $4-b$ while $\triangle QOP$ has legs b and b. Since the areas have a ratio of $9 : 25$, b must therefore satisfy the equation

$$\frac{(4-b)^2}{b^2} = \frac{9}{25} \Longrightarrow 16b^2 - 200b + 400 = 0 \Longrightarrow 2b^2 - 25b + 50 = 0$$

   which has roots $\frac{5}{2}$ and 10. Since $0 < b < 4$, this means that $b = \frac{5}{2}$.

9. If $\tan x + \tan y = 5$ and $\tan(x+y) = 10$, find $\cot^2 x + \cot^2 y$.

   Answer: 96

   Solution: We know that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = 10.$$

This leads to

$$\tan x \tan y = \frac{1}{2}.$$

Hence,

$$\begin{aligned}
\cot^2 x + \cot^2 y &= \frac{1}{\tan^2 x} + \frac{1}{\tan^2 y} \\
&= \frac{\tan^2 y + \tan^2 x}{\tan^2 x \tan^2 y} \\
&= \frac{(\tan x + \tan y)^2 - 2\tan x \tan y}{(\tan x \tan y)^2} \\
&= \frac{5^2 - 1}{\frac{1}{4}} \\
&= 96.
\end{aligned}$$

10. Let $\square ABCD$ be a trapezoid with parallel sides AB and CD of lengths 6 units and 8 units, respectively. Let E be the point of intersection of the extensions of the nonparallel sides of the trapezoid. If the area of $\triangle BEA$ is 60 square units, what is the area of $\triangle BAD$?

    Answer: 20

    Solution: Note that $\triangle BEA \sim \triangle CED$ and $|EB| = \frac{2 \cdot 60}{6} = 20$. Thus, $|EC| = \frac{8}{6} \cdot 20 = \frac{80}{3}$, and hence $|BC| = \frac{80}{3} - 20 = \frac{20}{3}$. Finally then,

    $$\text{area of } \triangle BAD = \frac{1}{2} \cdot 6 \cdot \frac{20}{3} = 20$$

11. How many solutions does the equation $x + y + z = 2016$ have, where x, y and z are integers with $x > 1000$, $y > 600$, and $z > 400$?

    Answer: 105

    First, note that the equation is equivalent to $(x-1001)+(y-601)+(z-401) = 13$. Letting $x' = x - 1001$, $y' = y - 601$, and $z' = z - 401$, we can instead count the number of solutions of $x' + y' + z' = 13$, where x', y', z' are nonnegative integers. Now note that each solution (x', y', z') to this equation corresponds to a way of choosing 13 items from a pile of objects of 3 types; the number of such choices is

    $$\binom{13+3-1}{13} = \frac{15!}{13!2!} = 105.$$

12. Find all values of integers x and y satisfying $2^{3x} + 5^{3y} = 189$.

    Answer: $x = 2$ and $y = 1$.

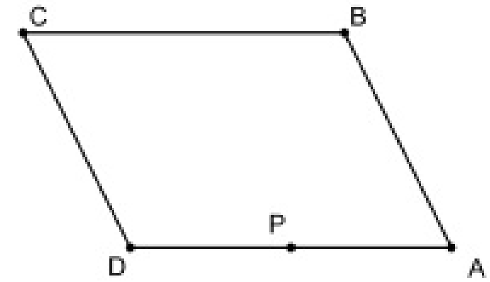
    Solution: We have

    $$\begin{aligned}
    2^{3x} + 5^{3y} &= (2^x + 5^y)(2^{2x} - 2^x 5^y + 5^{2y}) \\
    &= (9)(21) \\
    &= 189.
    \end{aligned}$$

    Clearly, $(x, y) = (2, 1)$ is the only solution pair to the equation.

13. In parallelogram $ABCD$, $\angle BAD = 76°$. Side AD has midpoint P, and $\angle PBA = 52°$. Find $\angle PCD$.

Answer: 38°



Solution: Note that $\angle BPA = 180° - 76° - 52° = 52°$. Since $\angle PBA = 52°$, then $\triangle BPA$ is isoceles and $|AB| = |AP| = |PD|$. But $|AB| = |CD|$, so by transitivity $|PD| = |CD|$ and therefore $\triangle PCD$ is also isoceles. Since $\angle CDA = 180° - 76° = 104°$, then $\angle PCD = \dfrac{180° - 104°}{2} = 38°$.

14. Find the smallest number k such that for all real numbers x, y and z

$$(x^2 + y^2 + z^2)^2 \leq k(x^4 + y^4 + z^4).$$

Answer: $k = 3$

Solution: Note that

$$(x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2.$$

Using the AM-GM Inequality, we find that

$$2x^2y^2 + 2x^2z^2 + 2y^2z^2 \leq 2[(x^4 + y^4)/2] + 2[(x^4 + z^4)/2] + 2[(y^4 + z^4)/2],$$

which when combined with the previous equation results to

$$(x^2 + y^2 + z^2)^2 \leq 3(x^4 + y^4 + z^4).$$

Therefore $k = 3$.

15. Last January 7, 2016, a team from the University of Central Missouri headed by Curtis Cooper discovered the largest prime number known so far:

$$2^{74,207,281} - 1$$

which contains over 22.3 million digits. Curtis Cooper is part of a large collaborative project called GIMPS, where mathematicians use their computers to look for prime numbers of the form 1 less than a power of 2. What is the meaning of GIMPS?

Answer: Great Internet Mersenne Prime Search

$\boxed{\text{AVERAGE}}$ 30 seconds, 3 points

1. Find the value of $\cot\left(\cot^{-1}2+\cot^{-1}3+\cot^{-1}4+\cot^{-1}5\right)$.

   Answer: $\dfrac{5}{14}$

   Solution: Using the identity
$$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right),$$
   the expression $\cot^{-1}2+\cot^{-1}3+\cot^{-1}4+\cot^{-1}5$ can be simplified to $\cot^{-1}\frac{9}{7}+\cot^{-1}\frac{11}{7}$. Thus, we have
$$\cot(\cot^{-1}2+\cot^{-1}3+\cot^{-1}4+\cot^{-1}5)=\cot\left(\cot^{-1}\frac{9}{7}+\cot^{-1}\frac{11}{7}\right)=\frac{5}{14}.$$

2. Find the minimum value of x^2+4y^2-2x, where x and y are real numbers that satisfy $2x+8y=3$.

   Answer: $-\dfrac{19}{20}$

   Solution: By the Cauchy-Schwarz Inequality,
$$[2(x-1)+4(2y)]^2\le(2^2+4^2)[(x-1)^2+4y^2].$$
   Now, with $2(x-1)+4(2y)=2x+8y-2=3-2=1$, we obtain
$$\begin{aligned}x^2+4y^2-2x &= (x-1)^2+4y^2-1\\ &\ge \frac{[2(x-1)+4(2y)]^2}{2^2+4^2}-1\\ &= \frac{1}{20}-1\\ &= -\frac{19}{20}.\end{aligned}$$
   The minimum $-\dfrac{19}{20}$ is indeed obtained with $x=\dfrac{11}{10}$ and $y=\dfrac{1}{10}$.

3. Alice, Bob, Charlie and Eve are having a conversation. Each of them knows who are honest and who are liars. The conversation goes as follows:

   Alice: Both Eve and Bob are liars.
   Bob: Charlie is a liar.
   Charlie: Alice is a liar.
   Eve: Bob is a liar.

   Who is/are honest?

   Answer: Charlie and Eve

   Solution: We consider two cases:
   Case 1: Alice is honest.
   If Alice is honest, both Eve and Bob must be liars. If Eve is a liar, then Bob must be honest. This cannot be the case.
   Case 2: Alice is a liar.
   If Alice is liar, then either Eve is honest or Bob is honest. Suppose Eve is honest. Then, Bob is a liar. If Bob is a liar, Charlie must be honest, and Alice is a liar. This is a possible case.

   Suppose Eve is a liar. Then, Bob is honest. Since Bob is honest, Charlie is a liar, and Alice is honest. This cannot also be the case.

   Thus the only possible case is that Alice is a liar, Bob is a liar, Charlie is honest and Eve is honest.

4. Let $f(x)$ be a polynomial function of degree 2016 whose 2016 zeroes have a sum of S. Find the sum of the 2016 zeroes of $f(2x-3)$ in terms of S.

<u>Answer</u>: $\dfrac{1}{2}S + 3024$

<u>Solution</u>: Let $r_1, r_2, \cdots, r_{2016}$ be the zeroes of $f(x)$. We can then write f as

$$f(x) = c(x - r_1)(x - r_2)\cdots(x - r_{2016}),$$

where $\displaystyle\sum_{i=1}^{2016} r_i = S$. Thus

$$f(2x-3) = c(2x - 3 - r_1)(2x - 3 - r_2)\cdots(2x - 3 - r_{2016})$$

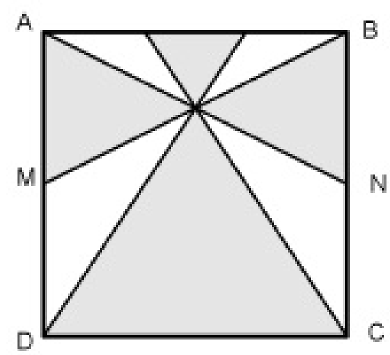
which has zeroes

$$\frac{r_1+3}{2}, \frac{r_2+3}{2}, \ldots, \frac{r_{2016}+3}{2}$$

This means that the required sum is

$$\sum_{i=1}^{2016} \frac{r_i+3}{2} = \frac{1}{2}\left[\sum_{i=1}^{2016} r_i + 3(2016)\right] = \frac{1}{2}S + 3024.$$

5. Refer to the figure on the right. The quadrilateral $ABCD$ is a square with a side of length 2 units while M and N are the midpoints of AD and BC, respectively. Determine the area of the shaded region.



<u>Answer</u>: $\dfrac{8}{3}$

<u>Solution</u>: The areas of the left and right triangles are both one-fourth the area of the area of the rectangle $ABNM$ which gives an area of $\frac{1}{2}$. The area of the lower triangle is $\frac{1}{2}(2)\left(\frac{3}{2}\right) = \frac{3}{2}$. The area of the upper triangle is $\frac{1}{9}$th the lower triangle (compare their altitudes). Hence, the area is $\frac{3}{2}\left(\frac{1}{9}\right)$.

Therefore, the area of the shaded region is

$$\frac{1}{2} + \frac{1}{2} + \frac{3}{2} + \frac{1}{6} = \frac{8}{3}.$$

6. Suppose that Ethan has four red chips and two white chips. He selects three chips at random and places them in Urn 1, while the remaining chips are placed in Urn 2. He then lets his brother Josh draw one chip from each urn at random. What is the probabiity that the chips drawn by Josh are both red?

   Answer: $\dfrac{2}{5}$

   Solution: The possibilities are

Urn 1	Urn 2	Number of Ways
1R, 2W	3R	$\binom{4}{1}\binom{2}{2} = 4$
2R, 1W	2R, 1W	$\binom{4}{2}\binom{2}{1} = 12$
3R	1R, 2W	$\binom{4}{3}\binom{2}{0} = 4$

   The total number of ways to split the chips into the two urns is twenty. Hence, the probability of getting both red chips is

$$\frac{4}{20}\left(\frac{1}{3}\right) + \frac{12}{20}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \frac{4}{20}\left(\frac{1}{3}\right) = \frac{2}{5}.$$

7. Let $f(x)$ be a function such that $f(1) = 1$, $f(2) = 2$ and $f(x+2) = f(x+1) - f(x)$. Find $f(2016)$.

   Answer: -1

   Solution:

$$\begin{array}{rcccccl}
f(1) & = & 1 & & & & \\
f(2) & = & 2 & & & & \\
f(3) & = & f(2) - f(1) & = & 2 - 1 & = & 1 \\
f(4) & = & f(3) - f(2) & = & 1 - 2 & = & -1 \\
f(5) & = & f(4) - f(3) & = & -1 - 1 & = & -2 \\
f(6) & = & f(5) - f(4) & = & -2 - (-1) & = & -1 \\
f(7) & = & f(6) - f(5) & = & -1 - (-2) & = & 1 \\
f(8) & = & f(7) - f(6) & = & 1 - (-1) & = & 2 \\
 & & \vdots & & & &
\end{array}$$

   Observe that this pattern will repeat itself every six, and thus, $f(2016) = f(6) = -1$.

8. In a certain school, there are 5000 students. Each student is assigned an ID number from 0001 to 5000. No two students can have the same ID number. If a student is selected uniformly at random, what is the probability that the ID number of the student does not contain any 2s among its digits?

   Answer: $\dfrac{729}{1250}$

   Solution: The number of ID numbers in the range 0001 to 5000 which do not contain any 2s is given by $4 \times 9 \times 9 + 1 - 1 = 2916$. For the thousands digit, we can use a number from 0 to 4 and for the rest we can use any digit except 2. We add one since 5000 was not counted and we subtract one since there is no ID number of 0000.

   Thus, the probability that the student ID number does not contain any 2 among its digits is $\dfrac{2916}{5000} = \dfrac{729}{1250}$.

9. 120 unit cubes are put together to form a rectangular prism whose six faces are then painted. This leaves 24 unit cubes without any paint. What is the surface area of the prism?

   Answer: 148

   Solution: Let the length, width and height of the rectangular prism made by the 24 cubes without paint be denoted by ℓ, w, h (necessarily positive integers), respectively. Then, those of the prism made by the 120 cubes have measures $\ell+2, w+2$ and $h+2$, respectively. Hence, $\ell wh = 24$ and $(\ell+2)(w+2)(h+2) = 120 = 2^3 \cdot 3 \cdot 5$. WLOG, assume that 5 divides $\ell + 2$.

   If $\ell + 2 = 5$, then $\ell = 3$ and we get $wh = 8$ and $(w+2)(h+2) = 24$. The values $\{w, h\} = \{4, 2\}$ satisfy the problem constraints. In this case, the surface area is $2((l+2)(w+2) + (w+2)(h+2) + (l+2)(h+2)) = 2(30 + 20 + 24) = 148$.

   If $\ell + 2 = 10$, then $\ell = 8$ and so $wh = 3$ and $(w+2)(h+2) = 12$. The former implies $\{w, h\} = \{1, 3\}$ which does not satisfy the latter constraint.

   If $\ell + 2 \geq 15$, then $\ell \geq 13$ and therefore, $24 \geq 13wh$. This forces $w = h = 1$ which makes $120 = (l+2)(w+2)(h+2)$ divisible by 9, contradiction.

10. Let m be the product of all positive integer divisors of $360,000$. Suppose the prime factors of m are $p_1, p_2, \ldots, p_k$, for some positive integer k, and $m = p_1^{e_1} p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$, for some positive integers $e_1, e_2, \ldots, e_k$. Find $e_1 + e_2 + \ldots + e_k$.

    Answer: 630

    Solution: Let $d(m)$ is the number of postive divisors of m. Since $360000 = 2^6 \cdot 3^2 \cdot 5^4$, we have $m = 360000^{\frac{d(m)}{2}}$. Thus,

$$e_1 + e_2 + e_3 = \frac{(6+2+4)}{2} \cdot [(6+1)(2+1)(4+1)] = 630.$$

$\boxed{\text{DIFFICULT}}$ 60 seconds, 6 points

1. The irrational number 0.123456789101112... is formed by concatenating, in increasing order, all the positive integers. Find the sum of the first 2016 digits of this number after the decimal point.

   Answer: 8499

   Solution: Note that all the one-digit and two-digit numbers take up a total of $9 + 180 = 189$ digits. Thus, we stop at $\dfrac{2016-189}{3} + 99 = 708$.

   From 0 to 99, there are 10 occurrences of 0 to 9 in the ones place, and ten iterations of 0 to 9 in the tens place. Thus, the total digit sum from 0 to 99 is 20(45)=900. From 100 to 699, there are 6 occurrences of 0 to 99, plus 100 iterations each of 0 to 6 (in the hundreds digit), which gives us a digit sum of 100(21)+6(900) = 7500. This gives the numbers 700, 701, 702, 703, 704, 705, 706, 707, 708. These have a sum of 99.

   Combining everything we have, we obtain a total digit sum of $900 + 7500 + 99 = 8499$.

2. Suppose $\frac{1}{2} \le x \le 2$ and $\frac{4}{3} \le y \le \frac{3}{2}$. Determine the minimum value of

$$\frac{x^3y^3}{x^6 + 3x^4y^2 + 3x^3y^3 + 3x^2y^4 + y^6}.$$

   Answer: $\dfrac{27}{1081}$

   Solution: Note that

$$\begin{aligned}
\frac{x^3y^3}{x^6 + 3x^4y^2 + 3x^3y^3 + 3x^2y^4 + y^6} &= \frac{x^3y^3}{(x^2+y^2)^3 + 3x^3y^3} \\
&= \frac{1}{\dfrac{(x^2+y^2)^3}{x^3y^3} + 3} \\
&= \frac{1}{\left(\dfrac{x^2+y^2}{xy}\right)^3 + 3} \\
&= \frac{1}{\left(\dfrac{x}{y} + \dfrac{y}{x}\right)^3 + 3}.
\end{aligned}$$

   Let $u = \dfrac{x}{y}$. We have $\dfrac{x}{y} + \dfrac{y}{x} = u + \dfrac{1}{u}$. To get the minimum value of the entire expression, we need to make $u + \dfrac{1}{u}$ as large as possible. We can do this by setting $x = \dfrac{1}{2}$ and $y = \dfrac{3}{2}$. The function $f(u) = u + \dfrac{1}{u}$ is increasing on $(1, +\infty)$. Therefore $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{1}{3} + 3 = \dfrac{10}{3}$ and the minimum is

$$\frac{1}{\left(\dfrac{x}{y} + \dfrac{y}{x}\right)^3 + 3} = \frac{1}{\left(\dfrac{10}{3}\right)^3 + 3} = \frac{27}{1081}.$$

3. In an $n \times n$ checkerboard, the rows are numbered 1 to n from top to bottom, and the columns are numbered 1 to n from left to right. Chips are to be placed on this board so that each square has a number of chips equal to the absolute value of the difference of the row and column numbers. If the total number of chips placed on the board is 2660, find n.

Answer: 20

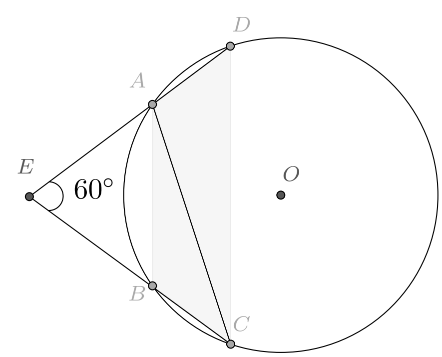
Solution: The total number of chips for an $n \times n$ board is equal to

$$\begin{aligned} n \times 0 + 2 \times (n-1) \times 1 + 2 \times (n-2) \times 2 + \cdots + 2 \times 1 \times (n-1) &= \sum_{i=1}^{n} 2 \times (n-i) \times i \\ &= 2\left(n \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2\right) \\ &= 2\left(n \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{1}{3}n^3 - \frac{1}{3}n. \end{aligned}$$

Find n such that $\frac{1}{3}n^3 - \frac{1}{3}n = 2660$. This leads to $n^3 - n = 7980$. Note that $n^3 - n - 7980 = (n-20)(n^2 + 20n + 399) = 0$ and the only integer satifying it is 20.

4. $ABCD$ is a cyclic quadrilateral such that $|DA| = |BC| = 2$, and $|AB| = 4$. If $|CD| > |AB|$ and the lines DA and BC intersect at an angle of $60°$, find the radius of the circumscribing circle.

Answer: $\dfrac{2\sqrt{21}}{3}$



Let E be the intersection of DA and BC. Since $|DA| = |BC|$, it follows from the power of a point formula that $|AE| = |BE|$, which in turn implies that ABE is an equilateral triangle. Hence, $\angle ABC = 120°$. Using cosine rule, $|AC| = \sqrt{20 - 16\cos 120°} = 2\sqrt{7}$. Therefore, by the extended law of sines, the circumdiameter $= \dfrac{|AC|}{\sin 120°} = \dfrac{2\sqrt{7}}{\sqrt{3}/2}$. This implies that the circumradius equals $2\sqrt{7/3}$ or $2\sqrt{21}/3$.

5. The faces of a 12-sided die are numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 such that the sum of the numbers on opposite faces is 13. The die is meticulously carved so that it is biased: the probability of obtaining a particular face F is greater than 1/12, the probability of obtaining the face opposite F is less than 1/12 while the probability of obtaining any one of the other ten faces is 1/12. When two such dice are rolled, the probability of obtaining a sum of 13 is 29/384. What is the probability of obtaining face F?

   Answer: 7/48

   Solution: The probabilities that the die lands on its corresponding faces are $\frac{1}{12}, \ldots, \frac{1}{12}, \frac{1}{12} + x, \text{and} \frac{1}{12} - x$, where the last two probabilities are for the face F and its opposite face, respectively, while the rest are for the other faces (since these probabilities must sum up to 1). Now, the sum of the results of the two dice can only be 13 if the results shown on both dice are such that they are opposites of one another. Hence,

$$10\left(\frac{1}{12}\right)\left(\frac{1}{12}\right) + 2\left(\frac{1}{12} + x\right)\left(\frac{1}{12} - x\right) = \frac{29}{384}.$$

   This leads to the quadratic equation

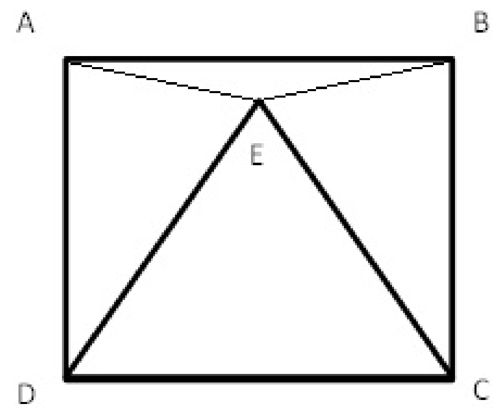
$$\frac{1}{144} - x^2 = \frac{7}{2304}.$$

   Solving for x, we get $x = \frac{1}{16}$ as the only plausible solution. Therefore, the probability of obtaining face F must be $\dfrac{1}{12} + \dfrac{1}{16} = \dfrac{7}{48}$.

$\boxed{\text{SPARE}}$ 30 seconds, 3 points

1. Inside square $ABCD$, a point E is chosen so that triangle DEC is equilateral. Find the measure of $\angle AEB$.

   Answer: $150°$

   Solution: Since $\triangle DEC$ is an equilateral triangle, then $|DE| = |CE|$ each angle has a measure of $60°$. This implies that $\angle ADE$ has measure of $30°$. Since $|AD| = |DE|$, then $\triangle ADE$ is an isosceles triangle. Thus, $\angle DAE = \angle DEA = 75°$. The same argument on triangle BEC will give us $\angle CBE = \angle CEB = 75°$. Thus, $\angle AEB = 150°$.

2. Find all triples of positive real numbers (x, y, z) which satisfy the system

$$\begin{cases} \sqrt[3]{x} - \sqrt[3]{y} - \sqrt[3]{z} = 64 \\ \sqrt[4]{x} - \sqrt[4]{y} - \sqrt[4]{z} = 32 \\ \sqrt[6]{x} - \sqrt[6]{y} - \sqrt[6]{z} = 8 \end{cases}$$

   Answer: no solution

   Solution: Using the first and the third equations, we find that

$$(8 + \sqrt[6]{y} + \sqrt[6]{z})^2 = 64 + \sqrt[3]{y} + \sqrt[3]{z}.$$

   Simplifying, we obtain the following equation

$$8\sqrt[6]{y} + 8\sqrt[6]{z} + \sqrt[6]{yz} = 0,$$

   which has no solution.

3. Find the minimum value of $x^2 + 4y^2 - 2x$, where x and y are real numbers that satisfy $2x + 8y = 3$.

   Answer: $-\dfrac{19}{20}$

   Solution: By the Cauchy-Schwarz Inequality,

$$[2(x-1) + 4(2y)]^2 \leq (2^2 + 4^2)[(x-1)^2 + 4y^2].$$

   Now, with $2(x-1) + 4(2y) = 2x + 8y - 2 = 3 - 2 = 1$, we obtain

$$\begin{aligned} x^2 + 4y^2 - 2x &= (x-1)^2 + 4y^2 - 1 \\ &\geq \frac{[2(x-1) + 4(2y)]^2}{2^2 + 4^2} - 1 \\ &= \frac{1}{20} - 1 \\ &= -\frac{19}{20}. \end{aligned}$$

   The minimum $-\dfrac{19}{20}$ is indeed obtained with $x = \dfrac{11}{10}$ and $y = \dfrac{1}{10}$.

4. Find all real numbers x that satisfies

$$\frac{x^4+x+1}{x^4+x^2-x-4} = \frac{x^4+1}{x^4+x^2-4}.$$

Answer: $-1, 0, 1$

Solution: Let $f(x) = x^4 + x + 1$ and $g(x) = x^4 + x^2 - x - 4$. The equation is then equivalent to

$$\frac{f(x)}{g(x)} = \frac{f(x)-x}{g(x)+x} \Longleftrightarrow x(f(x)+g(x)) = 0.$$

Hence, $x = 0$ or $f(x) + g(x) = 2x^4 + x^2 - 3 = (2x^2+3)(x^2-1) = 0$ while gives -1, 0, 1 as acceptable values of x.

5. Find all positive real numbers a, b, c, d such that for all $x \in \mathbb{R}$,

$$(ax+b)^{2016} + (x^2+cx+d)^{1008} = 8(x-2)^{2016}.$$

Answer: $a = 7^{\frac{1}{2016}}, b = -2 \cdot 7^{\frac{1}{2016}}, c = -4, d = 4$

Solution: Compare coefficients of x^{2016} in the equation to obtain $a^{2016} + 1 = 8$, i.e. $a = 7^{\frac{1}{2016}}$. Then, take $x = 2$ to obtain

$$(2a+b)^{2016} + (4+2c+d)^{1008} = 0.$$

Since the LHS is a sum of even-exponent powers, the equation will be solved in $\mathbb{R}$ if and only if both addends are zero. In particular, $b = -2a = -2 \cdot 7^{\frac{1}{2016}}$. Finally, substitute these values for a and b in the original equation to obtain

$$\begin{aligned} 7(x-2)^{2016} + (x^2+cx+d)^{1008} &= 8(x-2)^{2016} \\ (x^2+cx+d)^{1008} &= (x^2-4x+4)^{1008} \end{aligned}$$

Comparing coefficients once more, we finally obtain $c = -4$ and $d = 4$.

6. How many different integral solutions (x, y) does $3|x| + 5|y| = 100$ have?

Answer: 26

Solution: From $3|x| + 5|y| = 100$, we have $3|x| = 100 - 5|y| = 5(20 - |y|)$. This implies that $20 - |y|$ must be divisible by 3 and it must be nonnegative. Thus, $y = \pm 2, \pm 5, \pm 8, \pm 11, \pm 14, \pm 17, \pm 20$. Since each possible value of y has two possible x values except for $y = \pm 20$, then the number of ordered pairs is given by $12 \times 2 + 2 = 26$.

7. An ant situated at point A decides to walk 1 foot east, then $\frac{1}{2}$ foot northeast, then $\frac{1}{4}$ foot east, then $\frac{1}{8}$ foot northeast, then $\frac{1}{16}$ foot east and so on (that is, the ant travels alternately between east and northeast, and the distance travelled is decreased by half every time the ant changes its direction). The ant eventually reaches a certain point B. Determine the distance between the ant's unitial and final positions.

Answer: $\frac{2}{3}\sqrt{2\sqrt{2}+5}$ feet

Solution: The distance is the hypothenus of a right triangle. The length of its base is

$$\begin{aligned} 1 + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \frac{1}{8\sqrt{2}} + \frac{1}{16} + \frac{1}{32\sqrt{2}} + \cdots &= \left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) + \frac{1}{2\sqrt{2}}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= \frac{1}{1-\frac{1}{4}}\left(1 + \frac{1}{2\sqrt{2}}\right) \\ &= \frac{\sqrt{2}}{3} + \frac{4}{3}. \end{aligned}$$

Its height is

$$\begin{aligned}
\frac{1}{2\sqrt{2}} + \frac{1}{8\sqrt{2}} + \frac{1}{32\sqrt{2}} + \cdots &= \frac{1}{2\sqrt{2}}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\
&= \frac{1}{1-\frac{1}{4}}\left(\frac{1}{2\sqrt{2}}\right) \\
&= \frac{\sqrt{2}}{3}.
\end{aligned}$$

The distance is

$$\begin{aligned}
\sqrt{\left(\frac{\sqrt{2}}{3} + \frac{4}{3}\right)^2 + \frac{\sqrt{2}}{3})^2} &= \sqrt{\frac{8}{9}\sqrt{2} + \frac{20}{9}} \\
&= \frac{2}{3}\sqrt{2\sqrt{2} + 5}.
\end{aligned}$$

8. Find the last two digits of 2^{100}.

   Answer: 76

   Solution: Note that $2^{12} \equiv 96 \pmod{100} \equiv -4 \pmod{100}$. Thus, $2^{100} \equiv (2^{12})^8(2^4) \pmod{100} \equiv (-4)^8 2^4 \pmod{100} \equiv 2^{20} \pmod{100} \equiv (-4)2^8 \pmod{100} \equiv 76 \pmod{100}$

9. Let $f(x) = 2^x - 2^{1-x}$. Simplify $\sqrt{f(2015) - f(2014) + f(1) - f(0)}$.

   Answer: $2^{1007} + 2^{-1007}$

   Solution: We have $f(2015) = 2^{2015} - 2^{-2014}$, $f(2014) = 2^{2014} - 2^{-2013}$, $f(1) = 2 - 1 = 1$, and $f(0) = 1 - 2 = -1$. Hence,

$$\begin{aligned}
\sqrt{f(2015) - f(2014) + f(1) - f(0)} &= \sqrt{2^{2015} - 2^{-2014} - 2^{2014} + 2^{-2013} + 2} \\
&= \sqrt{(2^{2015} - 2^{-2014}) + (2^{-2013} - 2^{2014}) + 2} \\
&= \sqrt{2^{2014} + 2^{-2014} + 2} \\
&= \sqrt{(2^{1007} + 2^{-1007})^2} \\
&= 2^{1007} + 2^{-1007}.
\end{aligned}$$

10. A school program will randomly start between 8:30AM and 9:30AM and will randomly end between 7:00PM and 9:00PM. What is the probability that the program lasts for at least 11 hours and starts before 9:00AM?

    Answer: $\frac{5}{16}$

    Solution: Consider a rectangle R with diagonal having endpoints $(8.5, 19)$ and $(9.5, 21)$. Let S be the region inside R that is to the left of the line $x = 9$ and above the line $y = x + 11$. The desired probability is given by

$$\frac{\text{area of } S}{\text{area of } R} = \frac{\frac{5}{8}}{2} = \frac{5}{16}.$$

11. Find the last three digits of $2016^3 + 2017^3 + 2018^3 + \ldots + 3014^3$.

    Answer: 625

<u>Solution</u>: Note that $\{2016, 2017, 2018, \ldots, 3014, 3015\}$ comprise 1000 consecutive integers, and thus produce all possible residues modulo 1000. Thus,

$$\begin{aligned} 2016^3 + 2017^3 + 2018^3 + \ldots + 3014^3 + 3015^3 &\equiv 0^3 + 1^3 + 2^3 + \ldots + 999^3 \pmod{1000} \\ &= \left(\frac{999 \cdot 1000}{2}\right)^2 \\ &= 999^2 \cdot 500^2 \\ &= 999^2 \cdot 250000 \\ &\equiv 0 \pmod{1000} \end{aligned}$$

Therefore,

$$\begin{aligned} 2016^3 + 2017^3 + 2018^3 + \ldots + 3014^3 &= 2016^3 + 2017^3 + 2018^3 + \ldots + 3014^3 + 3015^3 - 3015^3 \\ &\equiv 0 - 3015^3 \pmod{1000} \\ &\equiv 1000 - 15^3 \pmod{1000} \\ &\equiv 1000 - 375 \pmod{1000} \\ &\equiv 625 \pmod{1000}. \end{aligned}$$