16^{th} Philippine Mathematical Olympiad

Area Stage

7 December 2013

Part I. No solution is needed. All answers must be in simplest form. Each correct answer is worth three points.

1. Find the number of ordered triples (x, y, z) of positive integers satisfying $(x+y)^z = 64$.
2. What is the largest number of $7\,\text{m} \times 9\,\text{m} \times 11\,\text{m}$ boxes that can fit inside a box of size $17\,\text{m} \times 37\,\text{m} \times 27\,\text{m}$?
3. Let $N = (1 + 10^{2013}) + (1 + 10^{2012}) + \cdots + (1 + 10^1) + (1 + 10^0)$. Find the sum of the digits of N.
4. The sequence $2, 3, 5, 6, 7, 8, 10, 11, \ldots$ is an enumeration of the positive integers which are not perfect squares. What is the 150th term of this sequence?
5. Let $P(x) = 1 + 8x + 4x^2 + 8x^3 + 4x^4 + \cdots$ for values of x for which this sum has finite value. Find $P(1/7)$.
6. Find all positive integers m and n so that for any x and y in the interval $[m, n]$, the value of $\dfrac{5}{x} + \dfrac{7}{y}$ will also be in $[m, n]$.
7. What is the largest positive integer k such that $27!$ is divisible by 2^k?
8. For what real values of p will the graph of the parabola $y = x^2 - 2px + p + 1$ be on or above that of the line $y = -12x + 5$?
9. Solve the inequality $\log\left(5^{\frac{1}{x}} + 5^3\right) < \log 6 + \log 5^{1+\frac{1}{2x}}$.
10. Let p and q be positive integers such that $pq = 2^3 \cdot 5^5 \cdot 7^2 \cdot 11$ and $\dfrac{p}{q} = 2 \cdot 5 \cdot 7^2 \cdot 11$. Find the number of positive integer divisors of p.
11. Let r be some real constant, and $P(x)$ a polynomial which has remainder 2 when divided by $x - r$, and remainder $-2x^2 - 3x + 4$ when divided by $(2x^2 + 7x - 4)(x - r)$. Find all values of r.
12. Suppose $\alpha, \beta \in (0, \pi/2)$. If $\tan\beta = \dfrac{\cot\alpha - 1}{\cot\alpha + 1}$, find $\alpha + \beta$.
13. How many positive integers, not having the digit 1, can be formed if the product of all its digits is to be 33750?
14. Solve the equation $(2 - x^2)^{x^2 - 3\sqrt{2}x + 4} = 1$.
15. Rectangle $BRIM$ has $BR = 16$ and $BM = 18$. The points A and H are located on IM and BM, respectively, so that $MA = 6$ and $MH = 8$. If T is the intersection of BA and IH, find the area of quadrilateral $MATH$.

16. Two couples and a single person are seated at random in a row of five chairs. What is the probability that at least one person is not beside his/her partner?

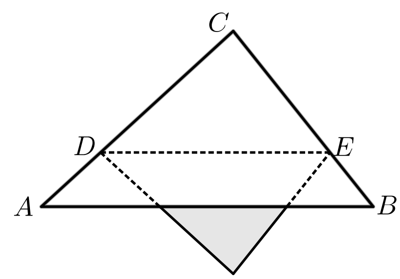
17. Trapezoid $ABCD$ has parallel sides AB and CD, with BC perpendicular to them. Suppose $AB = 13$, $BC = 16$ and $DC = 11$. Let E be the midpoint of AD and F the point on BC so that EF is perpendicular to AD. Find the area of quadrilateral $AEFB$.

18. Let x be a real number so that $x + \dfrac{1}{x} = 3$. Find the last two digits of $x^{2^{2013}} + \dfrac{1}{x^{2^{2013}}}$.

19. Find the values of x in $(0, \pi)$ that satisfy the equation

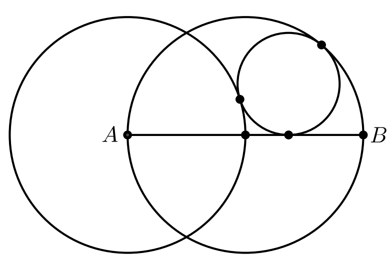
$$(\sqrt{2014} - \sqrt{2013})^{\tan^2 x} + (\sqrt{2014} + \sqrt{2013})^{-\tan^2 x} = 2(\sqrt{2014} - \sqrt{2013})^3.$$

20. The base AB of a triangular piece of paper ABC is 16 cm long. The paper is folded down over the base, with the crease DE parallel to the base of the paper, as shown. The area of the triangle that projects below the base (shaded region) is 16% that of the area of $\triangle ABC$. What is the length of DE, in cm?



Part II. Show the solution to each item. Each complete and correct solution is worth ten points.

1. Two circles of radius 12 have their centers on each other. As shown in the figure, A is the center of the left circle, and AB is a diameter of the right circle. A smaller circle is constructed tangent to AB and the two given circles, internally to the right circle and externally to the left circle, as shown. Find the radius of the smaller circle.



2. Let a, b and c be positive integers such that $\dfrac{a\sqrt{2013} + b}{b\sqrt{2013} + c}$ is a rational number. Show that $\dfrac{a^2 + b^2 + c^2}{a + b + c}$ and $\dfrac{a^3 - 2b^3 + c^3}{a + b + c}$ are both integers.

3. If p is a real constant such that the roots of the equation $x^3 - 6px^2 + 5px + 88 = 0$ form an arithmetic sequence, find p.

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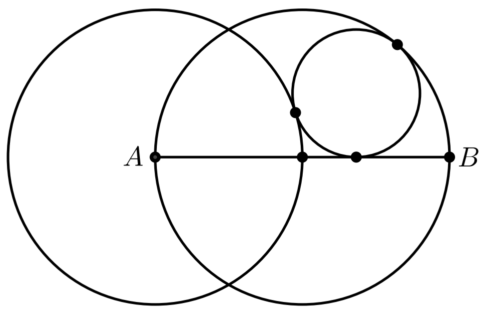
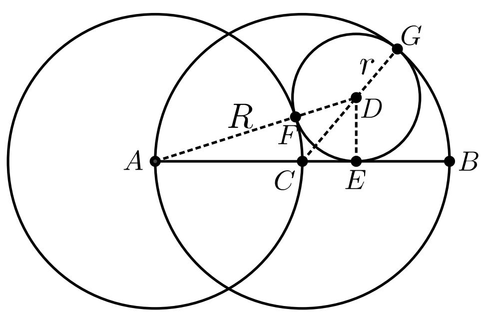
1. 74
2. 18
3. 2021
4. 162
5. $\frac{9}{4} = 2.25$
6. $(m, n) = (1, 12), (2, 6), (3, 4)$
7. 23
8. $5 \leq p \leq 8$
9. $\frac{1}{4} < x < \frac{1}{2}$
10. 72
11. $r = \frac{1}{2}, -2$
12. $\frac{\pi}{4}$
13. 625
14. $\pm 1, 2\sqrt{2}$
15. 34
16. $\frac{2}{5} = 0.4$
17. 91
18. 07
19. $x = \frac{\pi}{3}, \frac{2\pi}{3}$
20. 11.2 cm

Part II. Show the solution to each item. Each complete and correct solution is worth ten points.

1. Two circles of radius 12 have their centers on each other. As shown in the figure, A is the center of the left circle, and AB is a diameter of the right circle. A smaller circle is constructed tangent to AB and the two given circles, internally to the right circle and externally to the left circle, as shown. Find the radius of the smaller circle.



Solution:



Let R be the common radius of the larger circles, and r that of the small circle. Let C and D be the centers of the right large circle and the small circle, respectively. Let E, F and G be the points of tangency of the small circle with AB, the left large circle, and the right large circle, respectively. Since the centers of tangent circles are collinear with the point of tangency, then A-F-D and C-D-G are collinear.

From $\triangle AED$, $AE^2 = (R+r)^2 - r^2 = R^2 + 2Rr$. Therefore, $CE = AE - R = \sqrt{R^2 + 2Rr} - R$.

From $\triangle CED$, $CE^2 = (R-r)^2 - r^2 = R^2 - 2Rr$.

Therefore, $\sqrt{R^2 + 2Rr} - R = \sqrt{R^2 - 2Rr}$. Solving this for r yields $r = \frac{\sqrt{3}}{4}R$. With $R = 12$, we get $r = 3\sqrt{3}$.

2. Let a, b and c be positive integers such that $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c}$ is a rational number. Show that $\dfrac{a^2+b^2+c^2}{a+b+c}$ and $\dfrac{a^3-2b^3+c^3}{a+b+c}$ are both integers.

Solution:

By rationalizing the denominator, $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c} = \dfrac{2013ab-bc+\sqrt{2013}(b^2-ac)}{2013b^2-c^2}$. Since this is rational, then $b^2-ac=0$. Consequently,

$$\begin{aligned} a^2+b^2+c^2 &= a^2+ac+c^2 = (a+c)^2-ac = (a+c)^2-b^2 \\ &= (a-b+c)(a+b+c) \end{aligned}$$

and

$$\begin{aligned} a^3-2b^3+c^3 &= a^3+b^3+c^3-3b^3 = a^3+b^3+c^3-3abc \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \end{aligned}$$

Therefore,

$$\frac{a^2+b^2+c^2}{a+b+c} = a-b+c \qquad \text{and} \qquad \frac{a^3-2b^3+c^3}{a+b+c} = a^2+b^2+c^2-ab-bc-ca$$

are integers.

3. If p is a real constant such that the roots of the equation $x^3-6px^2+5px+88=0$ form an arithmetic sequence, find p.

Solution: Let the roots be $b-d$, b and $b+d$. From Vieta's formulas,

$$-88 = (b-d)b(b+d) = b(b^2-d^2) \tag{1}$$

$$5p = (b-d)b+b(b+d)+(b+d)(b-d) = 3b^2-d^2 \tag{2}$$

$$6p = (b-d)+b+(b+d) = 3b \tag{3}$$

From (3), $b=2p$. Using this on (1) and (2) yields $-44 = p(4p^2-d^2)$ and $5p = 12p^2-d^2$. By solving each equation for d^2 and equating the resulting expressions, we get $4p^2+\frac{44}{p} = 12p^2-5p$. This is equivalent to $8p^3-5p^2-44=0$. Since $8p^3-5p^2-44 = (p-2)(8p^2+11p+22)$, and the second factor has negative discriminant, we only have $p=2$.