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The PMO 2007-2008

Problems and Solutions of the Tenth Philippine Mathematical Olympiad

DOST-SEI • MSP • HSBC • VPHI

Preface and Introduction

This booklet contains the questions and answers in the Tenth Philippine Mathematical Olympiad, which was held during School Year 2007-2008.

First held in 1984, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: (1) to awaken greater interest in and promote the appreciation of mathematics among students and teachers; (2) to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; (3) to provide a vehicle for the professional growth of teachers; and (4) to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the International Mathematical Olympiad (IMO). It is followed by the Mathematical Olympiad Summer Camp (MOSC), a five-phase program for the twenty national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO is a continuing project of the Department of Science and Technology - Science Education Institute (DOST-SEI), and is being implemented by the Mathematical Society of the Philippines (MSP).

Though great effort was put in checking and editing the contents of this booklet, some errors may have slipped from the eyes of the reviewers. Should you find some errors, it would be greatly appreciated if these are reported to one of the authors at the following e-mail address:

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Quezon City — The Authors
20 June 2008

The Problems

Area Stage

24 November 2007

1. Simplify: $\left(\dfrac{2^{-1}+3^{-1}}{2^{-1}-3^{-1}}\right)^{-1}$.

2. If $2A99561$ is equal to the product when $3 \times (523 + A)$ is multiplied by itself, find the digit A.

3. The perimeter of a square inscribed in a circle is p. What is the area of the square that circumscribes the circle?

4. The sum of the first ten terms of an arithmetic sequence is 160. The sum of the next ten terms of the sequence is 340. What is the first term of the sequence?

5. It is given that $\triangle CAB \cong \triangle EFD$. If $AC = x + y + z$, $AB = z + 6$, $BC = x + 8z$, $EF = 3$, $DF = 2y - z$, and $DE = y + 2$, find $x^2 + y^2 + z^2$.

6. Container A contained a mixture that is 40% acid, while container B contained a mixture that is 60% acid. A chemist took some amount from each container, and mixed them. To produce 100 liters of mixture that is 17% acid, she needed to pour 70 liters of pure water to the mixture she got from containers A and B. How many liters did she take from container A?

7. If a and b are integers such that $a \log_{250} 2 + b \log_{250} 5 = 3$, what is the value of $a + 2b$?

8. Find all real values of x satisfying the inequality

$$\sqrt{\left(\frac{1}{2-x}+1\right)^2} \geq 2.$$

9. Find the polynomial of least degree, having integral coefficients and leading coefficient equal to 1, with $\sqrt{3}-\sqrt{2}$ as a zero.

10. Let $x=\cos\theta$. Express $\cos 3\theta$ in terms of x.

11. Solve the system of equations:

$$\begin{cases} x+y+\sqrt{xy} &= 28 \\ x^2+y^2+xy &= 336. \end{cases}$$

12. Let P be a point on the diagonal AC of the square $ABCD$. If AP is one-fourth of the length of one side of the square and the area of the quadrilateral $ABPD$ is 1 square unit, find the area of $ABCD$.

13. A circle is inscribed in $\triangle ABC$ with sides $AB=4$, $BC=6$, and $AC=8$. If P and Q are the respective points of tangency of $\overline{AB}$ and $\overline{AC}$ with the circle, determine the length of chord PQ.

14. If $\sqrt[3]{x+5}-\sqrt[3]{x-5}=1$, find x^2.

15. Let a, b, and c be real constants such that x^2+x+2 is a factor of ax^3+bx^2+cx+5, and $2x-1$ is a factor of $ax^3+bx^2+cx-\frac{25}{16}$. Find $a+b+c$.

16. Consider the function f defined by

$$f(x)=1+\frac{2}{x}.$$

Find the roots of the equation

$$(\underbrace{f\circ f\circ\cdots\circ f}_{10\text{ times}})(x)=x,$$

where "$\circ$" denotes composition of functions.

17. How many ordered pairs (x, y) of positive integers, where $x < y$, satisfy the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2007}.$$

18. Let ABC be an equilateral triangle. Let $\overrightarrow{AB}$ be extended to a point D such that B is the midpoint of $\overline{AD}$. A variable point E is taken on the same plane such that $DE = AB$. If the distance between C and E is as large as possible, what is $\angle BED$?

19. For what values of k does the equation

$$|x - 2007| + |x + 2007| = k$$

have $(-\infty, -2007) \cup (2007, +\infty)$ as its solution set?

20. Find the sum of the maximum and minimum values of

$$\frac{1}{1 + (2\cos x - 4\sin x)^2}.$$

21. Let k be a positive integer. A positive integer n is said to be a k-*flip* if the digits of n are reversed in order when it is multiplied by k. For example, 1089 is a 9-flip because $1089 \times 9 = 9801$, and 21978 is a 4-flip because $21978 \times 4 = 87912$. Explain why there is no 7-flip integer.

22. Let ABC be an acute-angled triangle. Let D and E be points on $\overline{BC}$ and $\overline{AC}$, respectively, such that $\overline{AD} \perp \overline{BC}$ and $\overline{BE} \perp \overline{AC}$. Let P be the point where $\overrightarrow{AD}$ meets the semicircle constructed outwardly on $\overline{BC}$, and Q the point where $\overrightarrow{BE}$ meets the semicircle constructed outwardly on $\overline{AC}$. Prove that $PC = QC$.

23. Two friends, Marco and Ian, are talking about their ages.

Ian says, "My age is a zero of a polynomial with integer coefficients."

Having seen the polynomial $p(x)$ Ian was talking about, Marco exclaims, "You mean, you are seven years old? Oops, sorry I miscalculated! $p(7) = 77$ and not zero."

"Yes, I am older than that," Ian's agreeing reply.

Then Marco mentioned a certain number, but realizes after a while that he was wrong again because the value of the polynomial at that number is 85.

Ian sighs, "I am even older than that number."

Determine Ian's age.

National Stage

Oral Competition

12 January 2008

15-Second Round

15.1. If

$$\begin{cases} wxy & = & 10 \\ wyz & = & 5 \\ wxz & = & 45 \\ xyz & = & 12 \end{cases}$$

what is $w + y$?

15.2. Simplify: $(x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$.

15.3. By how much does the sum of the first 15 positive odd integers exceed the sum of the first 10 positive even integers?

15.4. Solve for x: $16^{1/8} + x^{1/4} = \dfrac{23}{5 - \sqrt{2}}$.

15.5. The area of a trapezoid is three times that of an equilateral triangle. If the heights of the trapezoid and the triangle are both equal to $8\sqrt{3}$, what is the length of the median of the trapezoid?

15.6. If $\frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos 2x$ is an identity, what is the value of C?

15.7. If $ABCDEF$ is a regular hexagon with each side of length 6 units, what is the area of $\triangle ACE$?

15.8. Find the smallest positive integer x such that the sum of x, $x+3$, $x+6$, $x+9$, and $x+12$ is a perfect cube.

15.9. The length of one side of the square $ABCD$ is 4 units. A circle is drawn tangent to $\overline{BC}$ and passing through the vertices A and D. Find the area of the circle.

15.10. If $f(x+y) = f(x)\cdot f(y)$ for all positive integers x, y and $f(1) = 2$, find $f(2007)$.

15.11. It is given that $\triangle ABC \sim \triangle DEF$. If the area of $\triangle ABC$ is $\frac{3}{2}$ times that of $\triangle DEF$ and $AB = BC = AC = 2$, what is the perimeter of $\triangle DEF$?

15.12. For which real numbers x does the inequality

$$2\log_x\left(\frac{a+b}{2}\right) \leq \log_x a + \log_x b$$

hold for all positive numbers a and b?

15.13. In Figure 1, what part of $\triangle ABC$ is shaded?

15.14. In how many ways can the letters of the word SPECIAL be permuted if the vowels are to appear in alphabetical order?

15.15. Graph theory's Four-Color Theorem says that four colors are enough to color the regions in a plane so that no two adjacent regions receive the

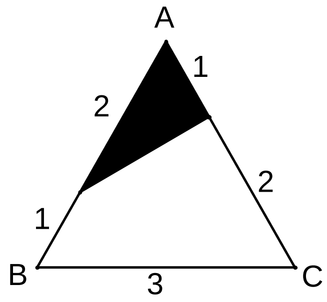


Figure 1: Problem 15.13.

same color. The theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken, 124 years after the Four-Color Problem was posed.

Fermat's Last Theorem in Number Theory was proved by Andrew Wiles in 1995, after 358 years of attempts by generations of mathematicians.

In 2003, Grigori Perelman completed the proof of a conjecture in topology. Considered as one of the seven millennium prize problems, the conjecture says that the sphere is the only type of bounded three-dimensional surface that contains no holes. Mathematicians worked on this conjecture for almost a century. What is the name of this conjecture that earned Perelman the Fields Medal which he refused to accept in 2006?

30-Second Round

30.1. What is the least 6-digit natural number that is divisible by 198?

30.2. Given that $x + 2$ and $x - 3$ are factors of $p(x) = ax^3 + ax^2 + bx + 12$, what is the remainder when $p(x)$ is divided by $x - 1$?

30.3. The graphs of $x^2 + y = 12$ and $x + y = 12$ intersect at two points. What is the distance between these points?

30.4. In an arithmetic sequence, the third, fifth and eleventh terms are distinct and form a geometric sequence. If the fourth term of the arithmetic sequence is 6, what is its 2007th term?

30.5. Let each of the characters A, B, C, D, E denote a single digit, and $ABCDE4$ and $4ABCDE$ represent six-digit numbers. If

$$4 \times ABCDE4 = 4ABCDE,$$

what is C?

30.6. Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the feet of the perpendiculars from B and C to $\overline{AC}$ and $\overline{AB}$, respectively. Suppose that $\overline{CE}$ and $\overline{BD}$ intersect at point H. If $EH = 1$ and $AD = 4$, find DE.

30.7. Find the number of real roots of the equation

$$4\cos(2007a) = 2007a.$$

30.8. In $\triangle ABC$, $\angle A = 15^\circ$ and $BC = 4$. What is the radius of the circle circumscribing $\triangle ABC$?

30.9. Find the largest three-digit number such that the number minus the sum of its digits is a perfect square.

30.10. The integer x is the least among three positive integers whose product is 2160. Find the largest possible value of x.

60-Second Round

60.1. Three distinct diameters are drawn on a unit circle such that chords are drawn as shown in Figure 2. If the length of one chord is $\sqrt{2}$ units and the other two chords are of equal lengths, what is the common length of these chords?

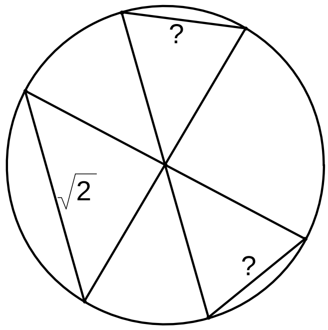


Figure 2: Problem 60.1.

60.2. If a and b are positive real numbers, what is the minimum value of the expression

$$\sqrt{a+b}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)?$$

60.3. What is the remainder when the sum

$$1^5 + 2^5 + 3^5 + \cdots + 2007^5$$

is divided by 5?

60.4. Let $ABCD$ be a square. Let M be the midpoint of $\overline{DC}$, N the midpoint of $\overline{AC}$, and P the intersection of $\overline{BM}$ and $\overline{AC}$. What is the ratio of the area of $\triangle MNP$ to that of the square $ABCD$?

60.5. Sharon has a chandelier containing n identical candles. She lights up the candles for n consecutive Sundays in the following manner: the first Sunday, she lights up one candle for one hour; the second Sunday, she lights up two candles, conveniently chosen, for one hour; and continues in the same fashion, increasing the number of candles lighted each Sunday by one, until in the nth Sunday, she lights up all the n candles for one hour. For what values of n is it possible for all the n candles to be of equal lengths right after the nth Sunday?

National Stage

Written Competition

12 January 2008

1. Prove that the set $\{1, 2, \ldots, 2007\}$ can be expressed as the union of disjoint subsets A_i $(i = 1, 2, \ldots, 223)$ such that

(a) each A_i contains 9 elements, and

(b) the sum of all the elements in each A_i is the same.

2. Find the largest integer n such that

$$\frac{n^{2007} + n^{2006} + \cdots + n^2 + n + 1}{n + 2007}$$

is an integer.

3. Let P be a point outside a circle, and let the two tangent lines through P touch the circle at A and B. Let C be a point on the minor arc AB, and let $\overrightarrow{PC}$ intersect the circle again at another point D. Let L be the line that passes through B and is parallel to $\overline{PA}$, and let L intersect $\overrightarrow{AC}$ and $\overrightarrow{AD}$ at points E and F, respectively. Prove that B is the midpoint of $\overline{EF}$.

4. Let f be the function defined by

$$f(x) = \frac{2008^{2x}}{2008 + 2008^{2x}}, \quad x \in \mathbb{R}.$$

Prove that

$$f\left(\frac{1}{2007}\right) + f\left(\frac{2}{2007}\right) + \cdots + f\left(\frac{2005}{2007}\right) + f\left(\frac{2006}{2007}\right) = 1003.$$

Answers and Solutions

Area Stage

1. $\dfrac{1}{5}$

$$\left(\frac{2^{-1}+3^{-1}}{2^{-1}-3^{-1}}\right)^{-1} = \frac{2^{-1}-3^{-1}}{2^{-1}+3^{-1}} = \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$$

2. 4

We are given that $2A99561 = [3 \times (523 + A)]^2$, which is equivalent to $2A99561 = 9 \times (523 + A)^2$. Since $(523 + A)^2$ is an integer, it follows that $2A99561$ is divisible by 9. By the rule on divisibility by 9, after adding all the digits of $2A99561$, it suffices to find the digit A for which $A + 5$ is divisible by 9, which yields $A = 4$.

3. $\dfrac{p^2}{8}$

The area of the square that circumscribes the circle is equal to the square of the diameter of the circle. The side of the inner square has length equal to $p/4$, so that the diameter of the circle (which is equal to the length of the diagonal of the inner square) is given by

$$\sqrt{\left(\frac{p}{4}\right)^2 + \left(\frac{p}{4}\right)^2} = \frac{\sqrt{2}p}{4}.$$

4. $\dfrac{79}{10}$

Let $a_1, a_2, \ldots, a_{20}$ be the arithmetic sequence, and let d be its common difference. Then $a_1 + a_2 + \cdots + a_{10} = 160$ and $a_1 + a_2 + \cdots + a_{10} + a_{11} + a_{12} + \cdots + a_{20} = 160 + 340 = 500$. Recalling the formula for the sum of an arithmetic series involving the first term a_1 and common difference d, the first equation yields $5(2a_1 + 9d) = 160$ or $2a_1 + 9d = 32$, while the second equation yields $10(2a_1 + 19d) = 500$ or $2a_1 + 19d = 50$. Thus, we get a system of linear equations:

$$\left\{ \begin{array}{rcl} 2a_1 + 9d & = & 32 \\ 2a_1 + 19d & = & 50. \end{array} \right.$$

Solving the system gives the value of a_1.

5. 21

Since $\triangle CAB \cong \triangle EFD$, it follows that $AC = EF$, $AB = FD$, and $BC = ED$. Thus, we need to solve the following system of linear equations:

$$\begin{cases} x+y+z &= 3 \\ z+6 &= 2y-z \\ x+8z &= y+2. \end{cases}$$

Solving the system gives $x = -2$, $y = 4$, and $z = 1$.

6. 5

Let a be the amount (in liters) of mixture the chemist took from container A, and b the amount she took from container B. Then $a + b + 70 = 100$. On the other hand, computing the amount of acid involved in the mixtures, we have $0.40a + 0.60b = 0.17(100)$ or $4a + 6b = 170$. Solving for a in the following system of equations:

$$\begin{cases} a+b+70 &= 100 \\ 4a+6b &= 170, \end{cases}$$

we get $a = 5$.

7. 21

Applying laws of logarithms to the given equation, we get

$$\log_{250}(2^a 5^b) = 3 \quad \text{or} \quad 2^a 5^b = 250^3 = 2^3 5^9.$$

Since a and b are integers and $\gcd(2,5) = 1$, we get $a = 3$ and $b = 9$, so that $a + 2b = 21$.

8. $[1, 2) \cup (2, 7/3]$

We recall that $\sqrt{a^2} = |a|$ for any $a \in \mathbb{R}$. Thus, the given inequality is equivalent to

$$\left| \frac{3-x}{2-x} \right| \geq 2,$$

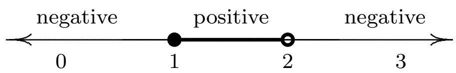
which is further equivalent to the following compound inequality:

$$\frac{3-x}{2-x} \geq 2 \quad \text{or} \quad \frac{3-x}{2-x} \leq -2. \tag{$\star$}$$

We solve the first inequality in $(\star)$.

$$\frac{3-x}{2-x} \geq 2 \quad \Longrightarrow \quad \frac{3-x}{2-x} - 2 \geq 0 \quad \Longrightarrow \quad \frac{x-1}{2-x} \geq 0$$

The last inequality gives $x = 1$ and $x = 2$ as critical numbers.



Thus, the solution set of the first inequality in $(\star)$ is $[1, 2)$.

Solving the second inequality in $(\star)$, we get the interval $(2, 7/3]$. Thus, the solution set of the original inequality is $[1, 2) \cup (2, 7/3]$.

9. $x^4 - 10x^2 + 1$

We let $x = \sqrt{3} - \sqrt{2}$. We find the monic polynomial equation of least degree in terms of x. Squaring, we get

$$x^2 = \left(\sqrt{3} - \sqrt{2}\right)^2 = 5 - 2\sqrt{6} \quad \text{or} \quad x^2 - 5 = -2\sqrt{6}.$$

Squaring the last equation, we finally get

$$(x^2 - 5)^2 = \left(-2\sqrt{6}\right)^2 \quad \text{or} \quad x^4 - 10x^2 + 1 = 0.$$

10. $4x^3 - 3x$

$$\begin{aligned}
\cos 3\theta &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= (2x^2 - 1)x - 2(1 - x^2)x \\
&= 4x^3 - 3x
\end{aligned}$$

11. $(4, 16)$ and $(16, 4)$

After rewriting the first equation into $x + y = 28 - \sqrt{xy}$, we square to get

$$x^2 + xy + y^2 = 784 - 56\sqrt{xy}.$$

Using the second given equation, the last equation becomes

$$336 = 784 - 56\sqrt{xy} \quad \text{or} \quad \sqrt{xy} = 8.$$

Substituting this last equation to the first given equation, we get $y = 20 - x$, and the second given equation then becomes

$$x^2 + (20 - x)^2 + 64 = 336,$$

which yields $x = 4$ and $x = 16$. Knowing that $xy = 64$, the solutions are the ordered pairs $(4, 16)$ and $(16, 4)$.

12. $4\sqrt{2}$ square units

Let s be the length of one side of the square $ABCD$. Let (ABP) and $(ABPD)$ denote the areas of $\triangle ABP$ and quadrilateral $ABPD$, respectively. Then $(ABP) = \frac{1}{2}(ABPD) = \frac{1}{2}$.

Since the diagonals of a square are perpendicular to each other and the length of each diagonal of $ABCD$ is equal to $\frac{1}{2}\sqrt{2}s$, we have

$$(ABP) = \frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}\left(\frac{s}{4}\right)\left(\frac{\sqrt{2}s}{2}\right) = \frac{\sqrt{2}s^2}{16}.$$

Since (ABP) is also equal to $\frac{1}{2}$, we get $s^2 = 4\sqrt{2}$ square units as the area of the square $ABCD$.

13. $\dfrac{3\sqrt{10}}{4}$

Applying the Law of Cosines to $\triangle ABC$, we get

$$\cos A = \frac{4^2 + 8^2 - 6^2}{2 \cdot 4 \cdot 8} = \frac{11}{16}.$$

Let $AP = AQ = x$, $PB = y$, and $QC = z$. Then we have the following system of equations:

$$\begin{cases} x+y &= 4 \\ x+z &= 8 \\ y+z &= 6. \end{cases}$$

Adding the equations, we get $2x + 2y + 2z = 18$ or $x + y + z = 9$, so that $x = (x+y+z) - (y+z) = 9 - 6 = 3$, $y = 1$, and $z = 5$. Finally, applying the Law of Cosines to $\triangle APQ$, we have

$$PQ^2 = AP^2 + AQ^2 - 2AP \cdot AQ \cos A = 3^2 + 3^2 - 2 \cdot 3 \cdot 3 \cdot \frac{11}{16} = \frac{45}{8}.$$

14. 52

By factoring with difference of two cubes, we have

$$\begin{aligned} &\left(\sqrt[3]{x+5}\right)^3 - \left(\sqrt[3]{x-5}\right)^3 \\ &= \left(\sqrt[3]{x+5} - \sqrt[3]{x-5}\right)\left(\sqrt[3]{(x+5)^2} + \sqrt[3]{(x+5)(x-5)} + \sqrt[3]{(x-5)^2}\right), \end{aligned}$$

which can be simplified into

$$(x+5) - (x-5) = \left(\sqrt[3]{(x+5)^2} + \sqrt[3]{x^2-25} + \sqrt[3]{(x-5)^2}\right)$$

or

$$\sqrt[3]{(x+5)^2} + \sqrt[3]{x^2-25} + \sqrt[3]{(x-5)^2} = 10. \qquad (\star)$$

On the other hand, squaring the given equation, we get

$$\sqrt[3]{(x+5)^2} - 2\sqrt[3]{x^2-25} + \sqrt[3]{(x-5)^2} = 1. \qquad (\star\star)$$

Subtracting $(\star\star)$ from $(\star)$, we obtain

$$3\sqrt[3]{x^2-25} = 9 \quad \text{or} \quad x^2 = 52.$$

15. $\dfrac{45}{11}$

Using long division, when ax^3+bx^2+cx+5 is divided by x^2+x+2, the quotient is $ax+(b-a)$ and the remainder is $(c-a-b)x+5+2a-2b$. Since x^2+x+2 is a factor of ax^3+bx^2+cx+5, we must have $c-a-b=0$ and $5+2a-2b=0$. On the other hand, since $2x-1$ is a factor of $ax^3+bx^2+cx-\frac{25}{16}$, by the Remainder Theorem, we must have

$$a\left(\frac{1}{2}\right)^3+b\left(\frac{1}{2}\right)^2+c\left(\frac{1}{2}\right)-\frac{25}{26}=0$$

or

$$\frac{a}{8}+\frac{b}{4}+\frac{c}{2}-\frac{25}{26}=0 \quad \text{or} \quad 2a+4b+8c-25=0.$$

Solving the following system of equations:

$$\begin{cases} c-a-b &= 0 \\ 5+2a-2b &= 0 \\ 2a+4b+8c-25 &= 0, \end{cases}$$

we get $a=-\frac{5}{22}$, $b=\frac{25}{11}$, and $c=\frac{45}{22}$, so that $a+b+c=\frac{45}{11}$.

16. -1 and 2

Let $f^{(n)}(x)=(\underbrace{f\circ f\circ\cdots\circ f}_{n\text{ times}})(x)$. For allowed values of x, note that $f^{(n)}(x)$ is of the form

$$f^{(n)}(x)=\frac{a_nx+b_n}{c_nx+d_n},$$

where $a_n,b_n,c_n,d_n\in\mathbb{Z}$ for all integers $n\geq 1$. The equation

$$\frac{a_nx+b_n}{c_nx+d_n}=x \quad \text{or} \quad c_nx^2+(d_n-a_n)x-b_n=0$$

has at most two real roots. Since $f(-1)=-1$ and $f(2)=2$, it follows that $f^{(n)}(-1)=-1$ and $f^{(n)}(2)=2$ for all $n\geq 1$. Thus, the roots of $f^{(10)}(x)=x$ are -1 and 2.

17. seven

We can rewrite the given equation into

$$(x-2007)(y-2007) = 2007^2 = 3^4 \cdot 223^2.$$

Since $x < y$, we have $x - 2007 < y - 2007$. It follows that

$$-2007 < x - 2007 < 2007 \quad \text{or} \quad |x - 2007| < 2007.$$

Thus, we have $|y - 2007| > 2007$.

$x-2007$	$y-2007$
1	$3^4 \cdot 223^2$
3	$3^3 \cdot 223^2$
3^2	$3^2 \cdot 223^2$
3^3	$3 \cdot 223^2$
3^4	223^2
223	$3^4 \cdot 223$
$3 \cdot 223$	$3^3 \cdot 223$

For every pair of values of $x - 2007$ and $y - 2007$ in the above table, there is a corresponding pair of x and y. Thus, there are seven such ordered pairs.

18. $15°$

To make C and E as far as possible, C, D, E must be collinear in that order.

With $\angle ABC = 60°$, we have $\angle CBD = 120°$. Since $BC = BD$, we then have $\angle CDB = \frac{1}{2}(180° - 120°) = 30°$. Finally, since $BD = DE$, we have $\angle BED = \frac{1}{2} \cdot 30° = 15°$.

19. $k > 4014$

If $x \in (-\infty, -2007)$, then

$$-(x-2007) - (x+2007) = k \quad \text{or} \quad x = -\frac{k}{2}.$$

If $x \in [-2007, 2007]$, then

$$-(x-2007)+(x+2007)=k \quad \text{or} \quad k=4014.$$

If $x \in (2007, +\infty)$, then

$$(x-2007)+(x+2007)=k \quad \text{or} \quad x=\frac{k}{2}.$$

Thus, the given equation has $(-\infty, -2007) \cup (2007, +\infty)$ as its solution set if and only if

$$\frac{k}{2} > 2007 \quad \text{or} \quad k > 4014.$$

20. $\dfrac{22}{21}$

Note that

$$2\cos x - 4\sin x = \sqrt{20}\left(\frac{2}{\sqrt{20}}\cos x - \frac{4}{\sqrt{20}}\sin x\right).$$

Let φ be a real number such that $\cos\varphi = \dfrac{2}{\sqrt{20}}$ and $\sin\varphi = \dfrac{4}{\sqrt{20}}$. We obtain

$$2\cos x - 4\sin x = \sqrt{20}\cos(x+\varphi).$$

Then

$$0 \leq (2\cos x - 4\sin x)^2 = 20\cos^2(x+\varphi) \leq 20.$$

Note here that we can particularly choose a value of x so that $20\cos^2(x+\varphi)=20$, and a value of x so that $20\cos^2(x+\varphi)=0$. Furthermore, we get

$$1 \leq 1+(2\cos x - 4\sin x)^2 \leq 21,$$

and so

$$\frac{1}{21} \leq \frac{1}{1+(2\cos x - 4\sin x)^2} \leq 1.$$

Thus, the sum of the maximum and minimum values of the given expression is $1+\frac{1}{21}=\frac{22}{21}$.

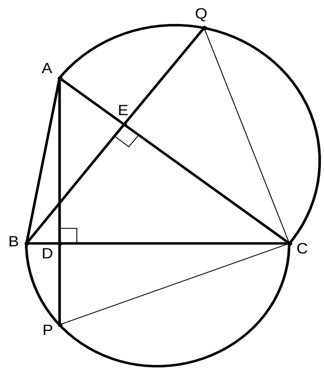


Figure 3: Problem 22.

21. Suppose, by way of contradiction, that the number $A \dots Z$ is a 7-flip. Then $A \dots Z \times 7 = Z \dots A$. So that there will be no carry-over in the multiplication of the last digit, A should be 1. This will imply two contradicting statements: (1) $Z \geq 7$ and (2) the product $7Z$ should have a units digit of 1, making Z equal to 3.

22. (This problem is taken from the British Mathematical Olympiad 2005.) Refer to Figure 3. By the Pythagorean Theorem, we have $QC^2 = EQ^2 + EC^2$. On the other hand, since $\triangle AQC$ is right-angled at Q and $\overline{QE} \perp \overline{AC}$, we have $EQ^2 = AE \cdot EC$. It follows that

$$QC^2 = EQ^2 + EC^2 = AE \cdot EC + EC^2 = EC(AE + EC) = EC \cdot AC.$$

Similarly, we also have

$$PC^2 = DC \cdot BC.$$

But since $\triangle ADC \sim \triangle BEC$, we obtain

$$\frac{DC}{AC} = \frac{EC}{BC} \quad \text{or} \quad DC \cdot BC = EC \cdot AC.$$

Thus, $PC^2 = QC^2$, which is equivalent to $PC = QC$.

23. Let a be Ian's age. Then

$$p(x) = (x - a)q(x),$$

where $q(x)$ is a polynomial with integer coefficients.

Since $p(7) = 77$, we have

$$p(7) = (7 - a)q(7) = 77 = 7 \cdot 11.$$

Since $q(7)$ is an integer and $7 - a < 0$, we restrict

$$a - 7 \in \{1, 7, 11, 77\}. \tag{$\star$}$$

Let b be the second number mentioned by Marco. Since $p(b) = 85$, we have

$$p(b) = (b - a)q(b) = 85 = 5 \cdot 17.$$

Since $q(b)$ is an integer and $b - a < 0$, we restrict

$$a - b \in \{1, 5, 17, 85\}. \tag{$\star\star$}$$

Finally, we know from algebra that $b - 7$ is a divisor of $p(b) - p(7) = 85 - 77 = 8 = 2^3$. It follows that

$$b - 7 \in \{1, 2, 4, 8\}. \tag{$\star\star\star$}$$

Considering all the possibilities from ($\star\star$) and ($\star\star\star$), since $a - 7 = (a - b) + (b - 7)$, we get

$$a - 7 \in \{2, 3, 5, 6, 7, 9, 13, 18, 19, 21, 25, 86, 87, 89, 93\}.$$

Recalling ($\star$), we get $a - 7 = 7$ or $a = 14$.

National Stage

Oral Competition

15.1. $\dfrac{19}{6}$

We multiply the four given equations.

$$(wxy)(wyz)(wxz)(xyz) = 10 \cdot 5 \cdot 45 \cdot 12$$

$$(wxyz)^3 = 2^3 3^3 5^3$$

$$wxyz = 2 \cdot 3 \cdot 5 = 30$$

$$w = \frac{wxyz}{xyz} = \frac{30}{12} = \frac{5}{2}, \quad y = \frac{wxyz}{wxz} = \frac{30}{45} = \frac{2}{3}$$

$$w + y = \frac{5}{2} + \frac{2}{3} = \frac{19}{6}$$

15.2. x^4

$$(x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1 = [(x-1)+1]^4 = x^4$$

15.3. 115

We use the formula for the sum of an arithmetic series.

$$\frac{15}{2}(2 + 14 \cdot 2) - \frac{10}{2}(4 + 9 \cdot 2) = 15^2 - 10 \cdot 11 = 115$$

15.4. 625

After rationalizing the denominator, we get

$$16^{1/8} + x^{1/4} = 5 + \sqrt{2}.$$

It follows that $x^{1/4} = 5$ or $x = 625$.

15.5. 24

The height of an equilateral triangle is $\dfrac{\sqrt{3}}{2}$ times the length of each of its sides. Thus, the length of one side of the equilateral triangle is $\dfrac{2}{\sqrt{3}}\left(8\sqrt{3}\right) = 16$, and its area is

$$\frac{1}{2}\left(8\sqrt{3}\right)(16) = 64\sqrt{3}.$$

Since the area of the trapezoid is three times that of the equilateral triangle, we have

$$\begin{aligned}(\text{height}) \times (\text{median of the trapezoid}) &= 3 \cdot 64\sqrt{3} \\ 8\sqrt{3} \times (\text{median of the trapezoid}) &= 3 \cdot 64\sqrt{3} \\ \text{median of the trapezoid} &= 24.\end{aligned}$$

15.6. $-\dfrac{1}{4}$

Since the equation is an identity, it is true for all x in the domain (which is $\mathbb{R}$) of the equation. To find C, we only set a particular value of x. For convenience, when we let $x = 0$, we have $C = -\frac{1}{4}$.

15.7. $27\sqrt{3}$ square units

Note that $\triangle ACE$ is equilateral. Each interior angle of $ABCDEF$ measures $\frac{1}{6}(6-2)(180°) = 120°$. Using a property of a 30°-60°-90° triangle, we have

$$\frac{1}{2}AC = \frac{\sqrt{3}}{2} \cdot 6 \quad \text{or} \quad AC = 6\sqrt{3}.$$

The height of $\triangle ACE$ is $\dfrac{\sqrt{3}}{2} \cdot 6\sqrt{3} = 9$, so that

$$\text{area of } \triangle ACE = \frac{1}{2} \cdot 6\sqrt{3} \cdot 9 = 27\sqrt{3}.$$

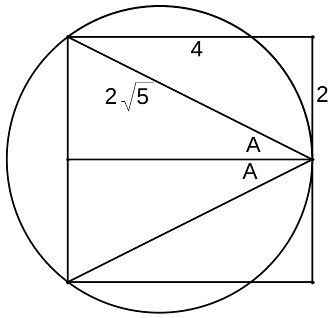


Figure 4: Problem 15.9.

15.8. 19

$$x + (x+3) + (x+6) + (x+9) + (x+12) = 5x + 30 = 5(x+6)$$

To make up the least possible cube, we must have $x+6 = 5^2$ or $x = 19$.

15.9. $\dfrac{25\pi}{4}$ square units

See Figure 4. Let R be the radius of the circle. Using the Extended Law of Sines, we have

$$2R = \frac{4}{\sin 2A} = \frac{4}{2\sin A\cos A} = \frac{2}{\frac{2}{2\sqrt{5}} \cdot \frac{4}{2\sqrt{5}}} = 5,$$

giving $R = \frac{5}{2}$. Thus, we get

$$\text{area of the circle} = \pi\left(\frac{5}{2}\right)^2 = \frac{25}{4}\pi.$$

15.10. 2^{2007}

By induction, it can be shown that $f(n) = [f(1)]^n = 2^n$ for all positive integers n. Thus, $f(2007) = 2^{2007}$.

15.11. $2\sqrt{6}$

We first define a notation. Let (XYZ) denote the area of $\triangle XYZ$.

$$\left(\frac{AB}{DE}\right)^2 = \frac{(ABC)}{(DEF)} = \frac{\frac{3}{2}(DEF)}{(DEF)} = \frac{3}{2} \implies \frac{AB}{DE} = \sqrt{\frac{3}{2}}$$

$$DE = \sqrt{\frac{2}{3}}\,AB = \frac{2\sqrt{6}}{3} = EF = DF$$

$$\text{perimeter of } \triangle DEF = 3 \cdot \frac{2\sqrt{6}}{3} = 2\sqrt{6}$$

15.12. $0 < x < 1$

The domain of the variable x in the inequality is $(0,1) \cup (1,+\infty)$. Use properties of logarithms, the inequality can be rewritten into

$$\log_x \left(\frac{a+b}{2}\right) \leq \log_x \sqrt{ab}.$$

By the AM-GM Inequality, we know that

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

It follows that the function $\log_x$ is decreasing, which only happens when $0 < x < 1$.

15.13. $\dfrac{2}{9}$

Refer to Figure 5. Triangle ABC is equilateral, which can be subdivided into 9 smaller equilateral triangles of side 1 unit in length. The shaded part comprises half the area of two smaller equilateral triangles. Thus, it is $\frac{2}{9}$ that of $\triangle ABC$.

15.14. 840

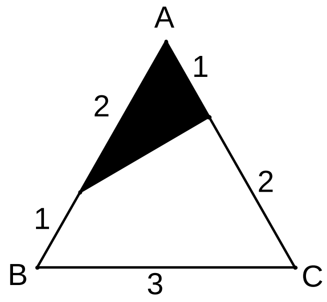


Figure 5: Problem 15.13.

We first arrange the letter without restrictions. There are 7! such arrangements. There are 3! ways to arrange the vowels into three particular positions, but only one of these is where the vowels are arranged in alphabetical order. Thus, the desired number of arrangements is $7! \div 3! = 4 \cdot 5 \cdot 6 \cdot 7 = 840$.

15.15. Poincaré Conjecture

30.1. 100188

Since $198 \times 500 = 99000$ and $198 \times 5 = 990$, we have $198 \times 505 = 99990$. Thus, the least six-digit natural number that is divisible by 198 is $99990 + 198 = 100188$.

30.2. 18

Since $x+2$ is a factor of $p(x) = ax^3 + ax^2 + bx + 12$, Factor Theorem guarantees that

$$p(-2) = -8a + 4a - 2b + 12 = 0 \quad \text{or} \quad 2a + b = 6. \tag{1$\star$}$$

Similarly, we also have

$$p(3) = 27a + 9a + 3b + 12 = 0 \quad \text{or} \quad 12a + b = -4. \tag{2$\star$}$$

Solving the system of equations involving (1$\star$) and (2$\star$), we get $a = -1$ and $b = 8$. Thus, we have $p(x) = -x^3 - x^2 + 8x + 12$, so that $p(1) = 18$.

30.3. $\sqrt{2}$

Solving the following system of equations using the elimination method:

$$\begin{cases} x^2 + y &= 12 \\ x + y &= 12, \end{cases}$$

we get the ordered pairs $(0, 12)$ and $(1, 11)$. That is, the given graphs intersect at $(0, 12)$ and $(1, 11)$, whose distance is $\sqrt{(0-1)^2 + (12-11)^2} = \sqrt{2}$.

30.4. 6015

Let a and d be the first term and the common difference of the given arithmetic sequence. Since there are distinct terms of the sequence, it follows that $d \neq 0$. Since the third, fifth and eleventh terms form a geometric sequence, we have

$$\frac{a+4d}{a+2d} = \frac{a+10d}{a+4d} \quad \text{or} \quad a + d = 0. \tag{$1\star$}$$

Since the fourth term of the sequence is 6, we also have

$$a + 3d = 6. \tag{$2\star$}$$

Solving the system of equations involving $(1\star)$ and $(2\star)$, we get $a = -3$ and $d = 3$. Thus, the 2007th term is $-3 + 2006(3) = 6015$.

30.5. 2

Let $x = ABCDE$. Then $4 \times ABCDE4 = 4ABCDE$ implies

$$4(10x + 4) = 400000 + x,$$

which gives us $x = 10256$, so that $C = 2$.

30.6. $\dfrac{8\sqrt{17}}{17}$

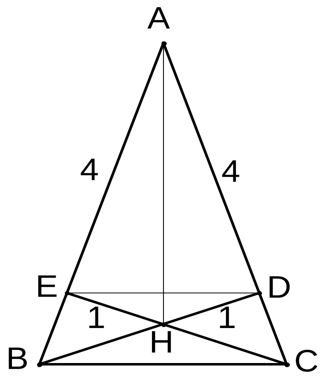


Figure 6: Problem 30.6.

Refer to Figure 6. By Pythagorean Theorem, we get $AH = \sqrt{17}$. Since $\angle AEH + \angle ADH = 90^\circ + 90^\circ = 180^\circ$, the quadrilateral $ADHE$ is cyclic. By Ptolemy's Theorem (in a cyclic quadrilateral, the product of the lengths of the diagonals equals the sum of the products of the lengths of the opposite sides), we have

$$DE \cdot \sqrt{17} = 1 \cdot 4 + 1 \cdot 4,$$

or $DE = \dfrac{8\sqrt{17}}{17}$.

30.7. three

Let $x = 2007a$. Then the given equation becomes $4\cos x = x$. The graphs of the equations $y = 4\cos x$ and $y = x$ intersect at three points. Thus, the equation $4\cos x = x$ has three roots. Consequently, the equation $4\cos(2007a) = 2007a$ also has three roots.

30.8. $4\sqrt{2+\sqrt{3}}$ or $2(\sqrt{6}+\sqrt{2})$

We use the Extended Law of Sines. Let R be radius of the circle circumscribing $\triangle ABC$.

$$2R = \frac{4}{\sin 15^\circ} \quad \text{or} \quad R = \frac{2}{\sin 15^\circ}$$

Depending on the formula used (either half-angle formula or difference of two angles), we have

$$\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2} \quad \text{or} \quad \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Then we solve for R using either of these values.

30.9. 919

Let abc be a three-digit number such that difference between the number and the sum of its digits is a perfect square; that is,

$$(100a + 10b + c) - (a + b + c) = 99a + 9b = 9(11a + b)$$

is a perfect square. To maximize the number $100a + 10b + c$, we set $a = 9$, $b = 1$, and $c = 9$.

30.10. 10

Note that $2160 = 2^4 \cdot 3^3 \cdot 5$. By trial-and-error method, we will notice that the set of three positive integers whose product is 2160 that will have a maximum least integer is $\{10, 12, 18\}$.

60.1. $\sqrt{2-\sqrt{2}}$ units

Refer to Figure 7. Let θ be the central angle subtended by the chord of length $\sqrt{2}$, and α the central angle subtended by each of the chords of equal lengths (and let x be this common length). By the Law of Cosines, we have

$$x^2 = 1^2 + 1^2 - 2\cos\alpha = 2 - 2\cos\alpha.$$

Since $\theta + 2\alpha = 180^\circ$, we get

$$\cos\alpha = \cos\left(90^\circ - \frac{\theta}{2}\right) = \sin\frac{\theta}{2} = \frac{\sqrt{2}}{2}.$$

We can then solve for x.

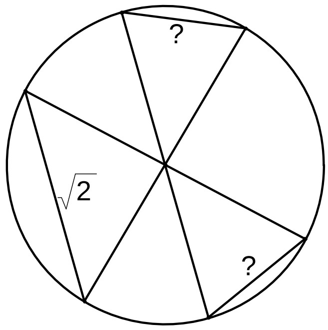


Figure 7: Problem 60.1.

60.2. $2\sqrt{2}$

By the AM-GM Inequality, we have

$$\sqrt{a+b} \geq \sqrt{2\sqrt{ab}} = \sqrt{2}(ab)^{1/4}$$

and

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \geq 2\sqrt{\frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}}} = \frac{2}{(ab)^{1/4}},$$

where both inequalities become equalities if and only if $a = b$. Multiplying the two inequalities, we get

$$\sqrt{a+b}\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) \geq 2\sqrt{2}.$$

60.3. 3

By Fermat's Little Theorem, we have $a^5 \equiv a \pmod 5$ for any integer a. Modulo 5, we have

$$1^5+2^5+3^5+\cdots+2007^5 \equiv 1+2+3+\cdots+2007 = 2007\cdot 1004 \equiv 2\cdot 4 \equiv 3.$$

Thus, the desired remainder is 3.

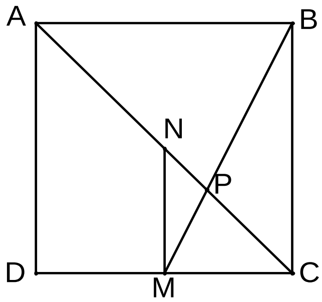


Figure 8: Problem 60.4.

60.4. 1 : 24

Refer to Figure 8. Notice that $\triangle MNP \sim \triangle BCP$, so that

$$\frac{NP}{PC} = \frac{MN}{BC} = \frac{1}{2} \quad \text{and} \quad \frac{MP}{BP} = \frac{MN}{BC} = \frac{1}{2}.$$

Recall that the ratio of the areas of two triangles of equal altitudes is equal to the ratio of the corresponding bases. With the notation (Z) to mean the area of polygon Z.

$$(MNP) = \frac{1}{2}(MPC) = \frac{1}{4}(BPC) = \frac{1}{4}\left[\frac{2}{3}(MBC)\right] = \frac{1}{6}\left[\frac{1}{4}(ABCD)\right],$$

and so $(MNP) : (ABCD) = 1 : 24$.

60.5. all positive odd integers

Since the candles are identical and they are of equal lengths right after the nth Sunday, they are used equal number of times for all the n Sundays. The total number of times they are used for the n Sundays is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

It follows that each candle is used $\frac{1}{2}(n+1)$ times for all the Sundays. Thus, since $\frac{1}{2}(n+1)$ must be an integer, n must be odd.

We need to exhibit a procedure to show that, when n is odd, it is indeed possible for Sharon to carry out the lighting of the candles. We first label the candles with the numbers $1, 2, \ldots, n$. We arrange these numbers in triangular array, writing $1, 2, \ldots, n$ consecutively, and goes back to 1 after n, until we complete the n Sundays. The procedure is illustrated below when $n = 7$:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 2 & & 3 & & & & & \\
 & & & & 4 & & 5 & & 6 & & & & \\
 & & & 7 & & 1 & & 2 & & 3 & & & \\
 & & 4 & & 5 & & 6 & & 7 & & 1 & & \\
 & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7
\end{array}$$

National Stage

Written Competition

1. We first arrange the numbers $670, 671, \ldots$ into 223 rows and 6 columns in the following way:

$$\begin{array}{ccccccccccc}
670 & & 1115 & \to & 1116 & & 1561 & \to & 1562 & & 2007 \\
671 & & 1114 & & 1117 & & 1560 & & 1563 & & 2006 \\
672 & & 1113 & & 1118 & & 1559 & & 1564 & & 2005 \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
890 & & 895 & & 1336 & & 1341 & & 1782 & & 1787 \\
891 & & 894 & & 1337 & & 1340 & & 1783 & & 1786 \\
892 & \to & 893 & & 1338 & \to & 1339 & & 1784 & \to & 1785
\end{array}$$

Let C_i represent the set containing the numbers in the ith row of the above arrangement. It is easy to check that the numbers in each C_i add up to a constant sum.

We now need to arrange the numbers $1, 2, \ldots, 669$ into 223 rows and 3 columns in such a way the sum of the numbers in each row is the same for all the rows:

$$\begin{array}{ccc}
1 & 335 & 669 \\
2 & 336 & 667 \\
3 & 337 & 665 \\
\downarrow & \downarrow & \downarrow \\
111 & 445 & 449 \\
112 & 446 & 447 \\
 & & \\
113 & 224 & 668 \\
114 & 225 & 666 \\
115 & 226 & 664 \\
\downarrow & \downarrow & \downarrow \\
221 & 332 & 452 \\
222 & 333 & 450 \\
223 & 334 & 448
\end{array}$$

Note that the sum of the numbers in the first and second columns of one row is different from the sum of the numbers in the first and second columns of another row. Since we expect that the sum of the numbers in each row is $(669)(670) \div (2)(223) = 1005$, we choose the number in the third column of a row to be the difference between 1005 and the sum of its numbers in the first and second columns. Thus, the numbers in the third column are distinct. Since the sum of the numbers in the first and second columns in each row ranges from 336 to 558, we expect that the numbers in the third column are the numbers from $1005 - 558 = 447$ to $1005 - 336 = 669$. Let B_i be the set containing the numbers in the ith row of the above arrangement.

The desired decomposition of the set $\{1, 2, \ldots, 2007\}$ is $A_i = B_i \cup C_i$, $i = 1, 2, \ldots, 223$.

2. Let $f(n) = n^{2007} + n^{2006} + \cdots + n^2 + n + 1$. As a geometric series, we have

$$f(n) = \frac{n^{2008} - 1}{n - 1}.$$

Using Division Algorithm, we can write

$$f(n) = \frac{n^{2008} - 1}{n - 1} = (n + 2007)g(n) + R,$$

where $g(n)$ is integer-valued function, and $R = f(-2007) = \dfrac{2007^{2008} - 1}{-2008}$, which is an integer, so that

$$f(n) = (n + 2007)g(n) - \frac{2007^{2008} - 1}{2008}.$$

So that $f(n)$ is divisible by $n + 2007$, we need $n + 2007$ to be a factor of R. To find the largest integer n, we should have

$$n + 2007 = \frac{2007^{2008} - 1}{2008} \quad \text{or} \quad n = \frac{2007^{2008} - 1}{2008} - 2007.$$

3. Refer to Figure 9. We only need to see four pairs of similar triangles.

$$\triangle AEB \sim \triangle ABC \quad \Longrightarrow \quad \frac{BE}{AB} = \frac{BC}{AC} \tag{1$\star$}$$

$$\triangle AFB \sim \triangle ABD \quad \Longrightarrow \quad \frac{BF}{AB} = \frac{BD}{AD} \tag{2$\star$}$$

$$\triangle PBC \sim \triangle PDB \quad \Longrightarrow \quad \frac{BC}{BP} = \frac{BD}{DP} \quad \Longrightarrow \quad \frac{BC}{BD} = \frac{BP}{DP} \tag{3$\star$}$$

$$\triangle PAC \sim \triangle PDA \quad \Longrightarrow \quad \frac{AC}{AP} = \frac{AD}{DP} \quad \Longrightarrow \quad \frac{AC}{AD} = \frac{AP}{DP} \tag{4$\star$}$$

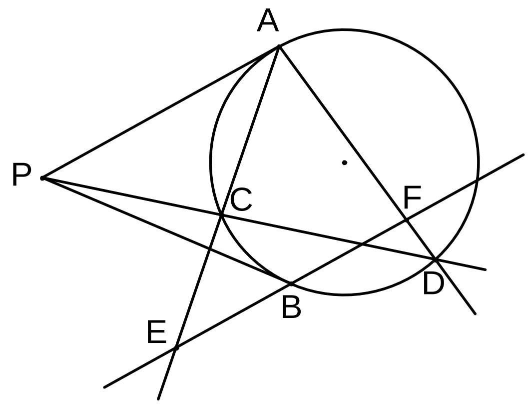


Figure 9: Problem 3.

Since $AP = BP$, from $(3\star)$ and $(4\star)$, we get

$$\frac{BC}{BD} = \frac{AC}{AD} \quad \text{or} \quad \frac{BC}{AC} = \frac{BD}{AD}.$$

Using this last proportion and applying transitivity property to $(1\star)$ and $(2\star)$ yield

$$\frac{BE}{AB} = \frac{BF}{AB} \quad \text{or} \quad BE = BF.$$

4. We first show that the function satisfies the identity $f(x)+f(1-x) = 1$.

$$f(1-x) = \frac{2008^{2(1-x)}}{2008 + 2008^{2(1-x)}} = \frac{2008^2 2008^{-2x}}{2008 + 2008^2 2008^{-2x}} = \frac{2008}{2008^{2x} + 2008}$$

$$f(x) + f(1-x) = \frac{2008^{2x}}{2008 + 2008^{2x}} + \frac{2008}{2008^{2x} + 2008} = 1$$

Pairing off the terms of the left-hand side of the desired equality into

$$\left[f\left(\frac{1}{2007}\right) + f\left(\frac{2006}{2007}\right)\right] + \cdots + \left[f\left(\frac{1003}{2007}\right) + f\left(\frac{1004}{2007}\right)\right],$$

and applying the above identity solve the problem.

10th Philippine Mathematical Olympiad

Area Stage: 24 November 2007
National Stage: 12 January 2008, UPNISMED

Project Director
Dr. Evangeline P. Bautista, *Ateneo de Manila University*

National Winners
`First Place`
Stephanie Anne A. Oliveros, *Philippine Science High School (Diliman)*

`Second Place`
Jillian Kristel G. Sy, *Chiang Kai Shek College*

`Third Place`
Diogo Miguel S. Moitinho de Almeida, *Ateneo de Manila High School*

49th International Mathematical Olympiad

10-22 July 2008, Madrid, Spain

Philippine IMO Team
`Contestants`
Jeffrey Kenneth L. Go, *Xavier School*
Diogo Miguel S. Moitinho de Almeida, *Ateneo de Manila High School*
Mark Benedict C. Tan, *Xavier School*

`Deputy Leader`
Dr. Julius M. Basilla, *University of the Philippines (Diliman)*

`Leader`
Dr. Ian June L. Garces, *Ateneo de Manila University*