22nd Philippine Mathematical Olympiad

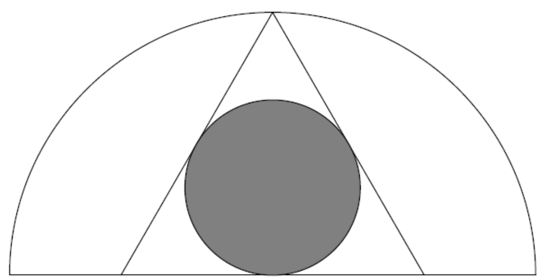
Area Stage, 16 November 2019

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. If the sum of the first 22 terms of an arithmetic progression is 1045 and the sum of the next 22 terms is 2013, find the first term.

2. How many positive divisors do 50,400 and 567,000 have in common?

3. In the figure below, an equilateral triangle of height 1 is inscribed in a semicircle of radius 1. A circle is then inscribed in the triangle. Find the fraction of the semicircle that is shaded.



4. Determine the number of ordered quadruples (a, b, c, d) of odd positive integers that satisfy the equation $a + b + c + d = 30$.

5. Suppose a real number $x > 1$ satisfies

$$\log_{\sqrt[3]{3}}(\log_3 x) + \log_3(\log_{27} x) + \log_{27}(\log_{\sqrt[3]{3}} x) = 1.$$

Compute $\log_3(\log_3 x)$.

6. Let $f(x) = x^2 + 3$. How many positive integers x are there such that x divides $f(f(f(x)))$?

7. In $\triangle XYZ$, let A be a point on (segment) YZ such that XA is perpendicular to YZ. Let M and N be the incenters of triangles XYA and XZA, respectively. If $YZ = 28$, $XA = 24$, and $YA = 10$, what is the length of MN?

8. Find the largest three-digit integer for which the product of its digits is 3 times the sum of its digits.

9. A wooden rectangular brick with dimensions 3 units by a units by b units is painted blue on all six faces and then cut into $3ab$ unit cubes. Exactly $1/8$ of these unit cubes have all their faces unpainted. Given that a and b are positive integers, what is the volume of the brick?

10. In square $ABCD$ with side length 1, E is the midpoint of AB and F is the midpoint of BC. The line segment EC intersects AF and DF at G and H, respectively. Find the area of quadrilateral $AGHD$.

11. A sequence $\{a_n\}_{n\geq 1}$ of positive integers satisfies the recurrence relation $a_{n+1} = n\lfloor a_n/n \rfloor + 1$ for all integers $n \geq 1$. If $a_4 = 34$, find the sum of all the possible values of a_1.

12. Let $(0,0)$, $(10,0)$, $(10,8)$, and $(0,8)$ be the vertices of a rectangle on the Cartesian plane. Two lines with slopes -3 and 3 pass through the rectangle and divide the rectangle into three regions with the same area. If the lines intersect above the rectangle, find the coordinates of their point of intersection.

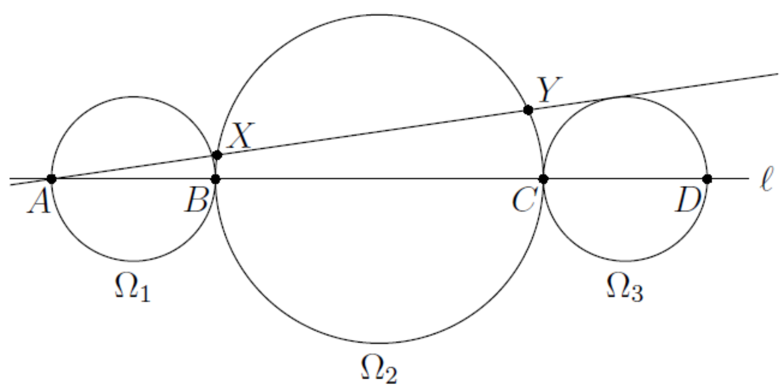
13. For a positive integer x, let $f(x)$ be the last two digits of x. Find $\displaystyle\sum_{n=1}^{2019} f\left(7^{7^n}\right)$.

14. How many positive rational numbers less than 1 can be written in the form $\dfrac{p}{q}$, where p and q are relatively prime integers and $p + q = 2020$?

15. The constant term in the expansion of $\left(ax^2 - \dfrac{1}{x} + \dfrac{1}{x^2}\right)^8$ is $210a^5$. If $a > 0$, find the value of a.

16. Let $A = \{n \in \mathbb{Z} \mid |n| \leq 24\}$. In how many ways can two distinct numbers be chosen (simultaneously) from A such that their product is less than their sum?

17. Points A, B, C, and D lie on a line ℓ in that order, with $AB = CD = 4$ and $BC = 8$. Circles Ω_1, Ω_2, and Ω_3 with diameters AB, BC, and CD, respectively, are drawn. A line through A and tangent to Ω_3 intersects Ω_2 at the two points X and Y. Find the length of XY.



18. A musical performer has three different outfits. In how many ways can she dress up for seven different performances such that each outfit is worn at least once? (Assume that outfits can be washed and dried between performances.)

19. In $\triangle PMO$, $PM = 6\sqrt{3}$, $PO = 12\sqrt{3}$, and S is a point on MO such that PS is the angle bisector of $\angle MPO$. Let T be the reflection of S across PM. If PO is parallel to MT, find the length of OT.

20. A student writes the six complex roots of the equation $z^6 + 2 = 0$ on the blackboard. At every step, he randomly chooses two numbers a and b from the board, erases them, and replaces them with $3ab - 3a - 3b + 4$. At the end of the fifth step, only one number is left. Find the largest possible value of this number.

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Consider all the subsets of $\{1, 2, 3, \ldots, 2018, 2019\}$ having exactly 100 elements. For each subset, take the greatest element. Find the average of all these greatest elements.

2. Let $a_1, a_2, \ldots$ be a sequence of integers defined by $a_1 = 3$, $a_2 = 3$, and

$$a_{n+2} = a_{n+1}a_n - a_{n+1} - a_n + 2$$

for all $n \geq 1$. Find the remainder when a_{2020} is divided by 22.

3. In $\triangle ABC$, $AB = AC$. A line parallel to BC meets sides AB and AC at D and E, respectively. The angle bisector of $\angle BAC$ meets the circumcircles of $\triangle ABC$ and $\triangle ADE$ at points X and Y, respectively. Let F and G be the midpoints of BY and XY, respectively. Let T be the intersection of lines CY and DF. Prove that the circumcenter of $\triangle FGT$ lies on line XY.

Answers to the 22nd PMO Area Stage

Part I. (3 points each)

1. $\frac{53}{2}$

2. 72

3. $\frac{2}{9}$

4. 560

5. $\frac{5}{13}$

6. 6

7. $2\sqrt{26}$

8. 951

9. 96

10. $\frac{7}{15}$

11. 130

12. $(5, 9)$

13. 50493

14. 400

15. $\frac{4}{3}$

16. 623

17. $\frac{24\sqrt{5}}{7}$

18. 1806

19. $2\sqrt{183}$

20. 730

Part II. (10 points each, full solutions required)

1. Consider all the subsets of $\{1, 2, 3, \ldots, 2018, 2019\}$ having exactly 100 elements. For each subset, take the greatest element. Find the average of all these greatest elements.

Solution 1: Let M be the average that we are computing. First, there are $\binom{2019}{100}$ ways to choose a 100-element subset. Next, if x is the largest element, then $x \geq 100$, and there are $\binom{x-1}{99}$ subsets having x as the largest element. Hence

$$M = \frac{\displaystyle\sum_{x=100}^{2019} x\binom{x-1}{99}}{\dbinom{2019}{100}}.$$

But note that

$$x\binom{x-1}{99} = 100\binom{x}{100}.$$

Using this fact and the hockey stick identity, we have

$$\begin{aligned} M &= \frac{100\displaystyle\sum_{x=100}^{2019}\binom{x}{100}}{\dbinom{2019}{100}} \\ &= \frac{100\dbinom{2020}{101}}{\dbinom{2019}{100}} \\ &= \frac{100 \cdot 2020}{101} \\ &= \boxed{2000} \end{aligned}$$

Solution 2: As in Solution 1, the required average M can be written as

$$M = \frac{\displaystyle\sum_{x=100}^{2019} x\binom{x-1}{99}}{\dbinom{2019}{100}} = \frac{100\dbinom{99}{99} + 101\dbinom{100}{99} + \cdots + 2019\dbinom{2018}{99}}{\dbinom{2019}{100}}.$$

We can simplify this expression by grouping the terms in the numerator in specific ways, applying the hockey stick identity to each group of terms, and then applying the hockey stick identity again to the resulting terms. This can be done in several ways, two of which are shown next:

Way 2.1: Note that

$$\begin{aligned}
\binom{2019}{100}M &= 100\binom{99}{99} + 101\binom{100}{99} + \cdots + 2019\binom{2018}{99} \\
&= 2020\left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] - \left[1920\binom{99}{99} + 1919\binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&= 2020\left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad - \left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad - \left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2017}{99}\right] \\
&\quad - \cdots - \binom{99}{99} \\
&= 2020\binom{2019}{100} - \binom{2019}{100} - \binom{2018}{100} - \cdots - \binom{100}{100} \\
&= 2020\binom{2019}{100} - \binom{2020}{101}.
\end{aligned}$$

where the last two lines follow from the hockey stick identity.

Hence, $M = 2020 - \dfrac{\dbinom{2020}{101}}{\dbinom{2019}{100}} = 2020 - \dfrac{2020}{101} = \boxed{2000}$.

Way 2.2: Let

$$X := \binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99} = \binom{2019}{100}.$$

Then

$$\begin{aligned}
\binom{2019}{100}M &= 100\binom{99}{99}+101\binom{100}{99}+\cdots+2019\binom{2018}{99} \\
&= 100\left[\binom{99}{99}+\binom{100}{99}+\cdots+\binom{2018}{99}\right] \\
&\quad+\left[\binom{100}{99}+\cdots+\binom{2018}{99}\right] \\
&\quad+\left[\binom{101}{99}+\cdots+\binom{2018}{99}\right] \\
&\quad+\cdots+\binom{2018}{99} \\
&= 100X+\left[X-\binom{100}{100}\right]+\left[X-\binom{101}{100}\right]+\cdots+\left[X-\binom{2018}{100}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
MX &= 100X+1919X-\left[\binom{100}{100}+\binom{101}{100}+\cdots+\binom{2018}{100}\right] \\
&= 2019X-\binom{2019}{101}
\end{aligned}$$

which gives

$$M=2019-\frac{\binom{2019}{101}}{\binom{2019}{100}}=2020-\frac{2020}{101}=\boxed{2000}.$$

2. Let $a_1, a_2, \ldots$ be a sequence of integers defined by $a_1 = 3,\ a_2 = 3$, and

$$a_{n+2} = a_{n+1}a_n - a_{n+1} - a_n + 2$$

for all $n \geq 1$. Find the remainder when a_{2020} is divided by 22.

Solution 1: Let $\{F_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$ be the sequence of Fibonacci numbers. We first claim that $a_n = 2^{F_n} + 1$ for all $n \in \mathbb{N}$. Clearly, this is true for $n = 1, 2$. Let $k \in \mathbb{N}$ and suppose that the claim is true for $n = k$ and for $n = k + 1$. Then

$$\begin{aligned}
a_{k+2} &= a_{k+1}a_k - a_{k+1} - a_k + 2 \\
&= (a_{k+1}-1)(a_k-1)+1 \qquad (1) \\
&= 2^{F_{k+1}}2^{F_k}+1 \\
&= 2^{F_{k+2}}+1.
\end{aligned}$$

By strong induction, the claim is proved. Therefore, we now find the remainder when $2^{F_{2020}} + 1$ is divided by 22. To this end, it is easier to find residues modulo 2 and modulo 11 and process them to get the residue modulo 22 (e.g. through Chinese Remainder Theorem).

Clearly, $2^{F_{2020}}$ is even, i.e., $2^{F_{2020}}+1 \equiv 1 \pmod 2$. We will see later that $2^{F_{2020}}+1 \equiv 0 \pmod{11}$. Therefore, by Chinese Remainder Theorem, $\boxed{a_{2020} = 2^{F_{2020}}+1 \equiv 11 \pmod{22}}$.

There are several ways to find the residue of $2^{F_{2020}}+1 \pmod{11}$.

Way 1.1: By Fermat's Little Theorem, $2^{10} \equiv 1 \pmod{11}$. This prompts us to consider the sequence of residues of $F_n \pmod{10}$ in order to find $F_{2020} \pmod{10}$:

$$\{F_n \pmod{10}\}_{n=1}^{\infty} = \{1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6,1,7,8,5,3,8,1,9,0,9,9,8,\\ 7,5,2,7,9,6,5,1,6,7,3,0,3,3,6,9,5,4,9,3,2,5,7,2,9,1,0,...cycle\}$$

We see that the sequence is cyclic with period 60. Therefore, since $2020 = 60(33)+40$, we obtain $F_{2020} \equiv 5 \pmod{10}$. Consequently, for some $k \in \mathbb{Z}$,

$$2^{F_{2020}}+1 = 2^{10k+5}+1 \equiv \left(2^{10}\right)^k 2^5+1 \equiv 33 \equiv 0 \pmod{11}. \tag{2}$$

Way 1.2: Another way to find $F_{2020} \pmod{10}$ is to get the residues of F_{2020} modulo 2 and 5 and process them to find the residue modulo 10. Again, we list down the sequence of residues modulo 2 and 5: $\{F_n \pmod 2\}_{n=1}^{\infty} = \{1,1,0,...cycle\}$ which has period 3, and

$$\{F_n \pmod 5\}_{n=1}^{\infty} = \{1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,0,...cycle\},$$

which has period 20. Since, $2020 \equiv 1 \pmod 3$ and $2020 \equiv 0 \pmod{20}$, then $F_{2020} \equiv 1 \pmod 2$ and $F_{2020} \equiv 0 \pmod 5$. Therefore, $F_{2020} \equiv 5 \pmod{10}$ by Chinese Remainder Theorem. Thus, (2) holds.

Solution 2: Inspired by the factorization in (1), we define $b_n = a_n - 1$ for all $n \in \mathbb{N}$. Then $b_1 = b_2 = 2$ and (1) simplifies to

$$b_{n+2} = b_{n+1}b_n. \tag{3}$$

We observe that $b_1 = 2^{F_1}$, $b_2 = 2^{F_2}$, and (3) implies $b_3 = 2^{F_3}$. This pattern continues and obviously shows that $b_n = 2^{F_n}$ for all $n \in \mathbb{N}$. Very similar arguments to Solution 1 will give $b_{2020} \equiv 0 \pmod 2$ and $b_{2020} \equiv 10 \pmod{11}$. Therefore, by Chinese Remainder Theorem, $b_{2020} \equiv 10 \pmod{22}$. Equivalently, $a_{2020} \equiv 11 \pmod{22}$.

Solution 3: By bashing, we can list down the residues of $\{a_n\}_{n=1}^{\infty}$ (or of $\{b_n\}_{n=1}^{\infty}$ as defined in Solution 2):

$$\{a_n \pmod{22}\}_{n=1}^{\infty} = \{3,3,5,9,11,15,9,3,17,11,7,17,9,19,13,19,19,17,3,11,21,3,19,15,\\ 11,9,15,3,7,13,7,7,15,19,11,5,19,7,21,11,3,21,19,9,13,9,9,21,7,11,\\ 17,7,9,5,11,19,5,7,3,13,...cycle\}.$$

Since the sequence is cyclic with period 60, and $2020 \equiv 40 \pmod{60}$, then $a_{2020} \equiv 11 \pmod{22}$.

3. In $\triangle ABC$, $AB = AC$. A line parallel to BC meets sides AB and AC at D and E, respectively. The angle bisector of $\angle BAC$ meets the circumcircle of $\triangle ABC$ and $\triangle ADE$ at points X and Y, respectively. Let F and G be the midpoints of BY and XY, respectively. Let T be the intersection of lines CY and DF. Prove that the circumcenter of $\triangle FGT$ lies on line XY.

Solution 1: Let F' be the reflection of F over line XY. Observe that, by symmetry, we get $\angle ADY = \angle AEY$. As quadrilateral $ADYE$ is cyclic, both angles must be right, and hence $\angle BDY$ is right as well.

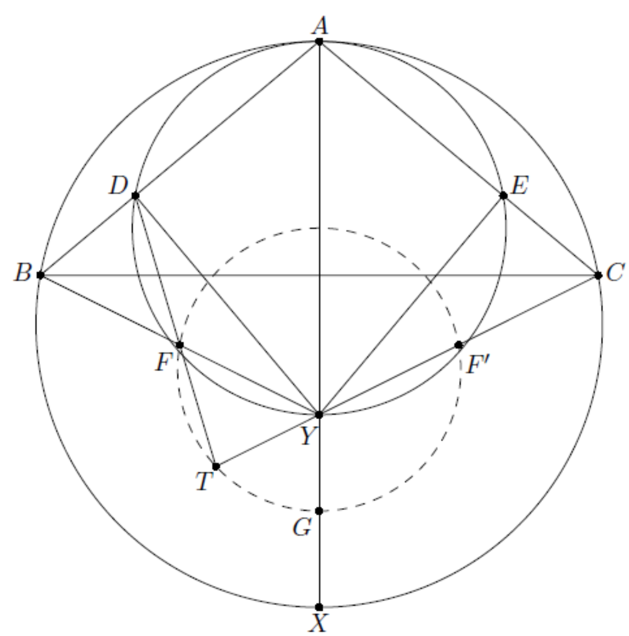
Thus F is the circumcenter of triangle BDY, so $\angle FDB = \angle FBD$. By reflecting over XY, we also get $\angle FBD = \angle F'CE$. Thus

$$\measuredangle TCA = \measuredangle F'CE = \measuredangle DBF = \measuredangle FDB = \measuredangle TDA,$$

so D, A, T, and C are concyclic. This implies that

$$\measuredangle FTF' = \measuredangle DTC = \measuredangle DAC = \measuredangle BAC = \measuredangle BXC = \measuredangle FGF',$$

the last step following from $FG \parallel BX$ and $F'G \parallel CX$. Thus the points F, T, G, and F' are concyclic. Their circumcenter must lie on the perpendicular bisector of FF', which is line XY. But this is also the circumcenter of triangle FGT, as desired.



Solution 2: As in Solution 1, AY and AX are diameters. It follows that

$$\angle ADY = \angle ABX = 90^\circ \implies BX \perp BD \perp DY.$$

Hence $BX \parallel DY$. As FG is a midline of $\triangle BXY$, it follows that it is the perpendicular bisector of BD. Then

$$\measuredangle GFT = \measuredangle GFD = \measuredangle BFG = \measuredangle YFG = \measuredangle GF'Y = \measuredangle GF'T,$$

which implies that $FF'GT$ is cyclic. The logic in Solution 1 finishes the proof.

Remark: The directed angles are necessary here due to configuration issues.