22^{nd} Philippine Mathematical Olympiad

National Stage, Written Phase

18 January 2020

Time: 4.5 hours *Each item is worth 7 points.*

1. A *T-tetromino* is formed by adjoining three unit squares to form a 1×3 rectangle, and adjoining on top of the middle square a fourth unit square. Determine the minimum number of unit squares that must be removed from a 202×202 grid so that it can be tiled with T-tetrominoes.

2. Determine all positive integers k for which there exist positive integers r and s that satisfy the equation

$$(k^2 - 6k + 11)^{r-1} = (2k - 7)^s.$$

3. Define the sequence $\{a_i\}$ by $a_0 = 1$, $a_1 = 4$, and $a_{n+1} = 5a_n - a_{n-1}$ for all $n \geq 1$. Show that all terms of the sequence are of the form $c^2 + 3d^2$ for some integers c and d.

4. Let ABC be an acute triangle with circumcircle Γ and D the foot of the altitude from A. Suppose that $AD = BC$. Point M is the midpoint of DC, and the bisector of $\angle ADC$ meets AC at N. Point P lies on Γ such that lines BP and AC are parallel. Lines DN and AM meet at F, and line PF meets Γ again at Q. Line AC meets the circumcircle of $\triangle PNQ$ again at E. Prove that $\angle DQE = 90°$.

22^{nd} Philippine Mathematical Olympiad

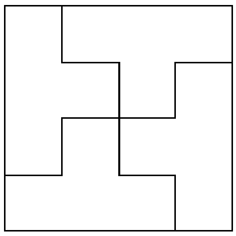
National Stage, Written Phase (Solutions)

18 January 2020

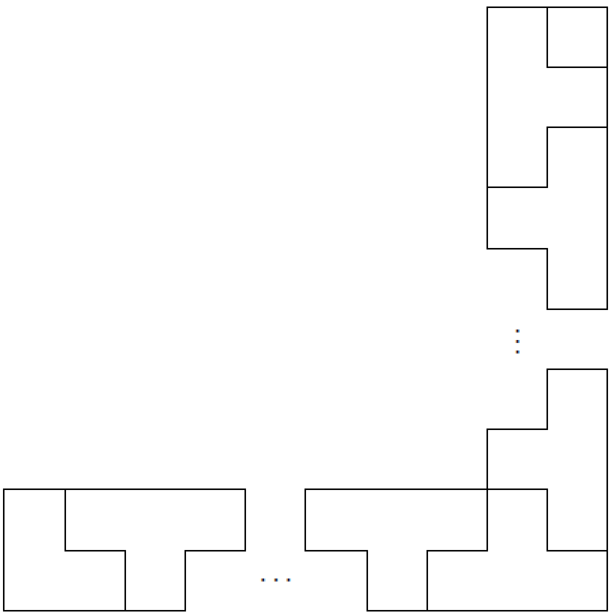
Time: 4.5 hours *Each item is worth 7 points.*

1. A *T-tetromino* is formed by adjoining three unit squares to form a 1×3 rectangle, and adjoining on top of the middle square a fourth unit square. Determine the minimum number of unit squares that must be removed from a 202×202 grid so that it can be tiled with T-tetrominoes.

<u>*Solution.*</u> We provide the following construction, which shows that the answer is at most four. Clearly, four T-tetrominoes tile a 4×4 grid, as follows:



These can be used to tile the upper-left 200×200 grid. It can be verified that the following pattern fills the remaining space, leaving four grid squares uncovered:



As the answer must be a multiple of four, it remains to prove that the answer is not zero, that is, it is impossible to tile a 202×202 grid with T-tetrominoes.

Number the rows of the grid, from left to right, with $0, 1, \ldots, 201$. Similarly, number the columns of the grid, from top to bottom, with $0, 1, \ldots, 201$.

Assign the square numbered (x, y) with the weight $4x + 4y + 1$. It can be verified that a T-tetromino, no matter how it is placed, covers a sum of weights divisible by 8. However, the sum of all the weights is not divisible by 8, as desired. ■

Solution 2: An alternative way to prove that the 202×202 grid cannot be tiled with T-tetrominoes is with a checkerboard coloring.

For the sake of contradiction, assume that the board can be tiled with T-tetrominoes. Clearly, a T-tetromino covers only either three or one black square. Let x be the number of T-tetrominoes each covering three black squares and y be the number of T-tetrominoes each covering only one black square. Counting the number of black and white squares yields

$$3x + y = 2 \cdot 101^2, \quad x + 3y = 2 \cdot 101^2 \quad \Longrightarrow \quad x = y = \frac{101^2}{2}.$$

which contradicts the fact that x and y are integers. ■

2. Determine all positive integers k for which there exist positive integers r and s that satisfy the equation

$$(k^2 - 6k + 11)^{r-1} = (2k - 7)^s$$

Solution: Clearly, if $r = 1$, then $2k - 7 = 1$ or $2k - 7 = -1$. Thus, two solutions are $k = 4$ and $k = 3$. Furthermore, notice that if $k = 2$, then $3^{r-1} = (-3)^s$, which has a solution for r and s. Thus, another solution is $k = 2$.

For $r \geq 2$, notice that $k^2 - 6k + 11 = (k-3)^2 + 2 \geq 2$ and $k^2 - 6k + 11 > 2k - 7$ because $(k-4)^2 > -2$. Moreover, $k^2 - 6k + 11$ and $(2k - 7)$ have the same prime factors. Let p be a prime factor of $k^2 - 6k + 11$ and $2k - 7$, then

$$p \mid [(k^2 - 6k + 11) + (2k - 7)] = (k-2)^2, \text{ which implies that } p \mid (k-2)$$

Moreover, $p \mid [2(k-2) - (2k-7)] = 3$. Hence, $p = 3$. This means that there are positive integers m and n with $m \geq n$ such that $k^2 - 6k + 11 = 3^m$ and $2k - 7 = 3^n$. Notice that

$$4 \cdot 3^m = 4(k-3)^2 + 8 = (2k-6)^2 + 8 = (3^n + 1)^2 + 8 = 3^{2n} + 2 \cdot 3^n + 9.$$

Since $3^n \mid (3^{2n} + 2 \cdot 3^n + 9)$, then $3^n|9$ and $n \leq 2$. Hence, we only have two cases left.

- If $n = 1$, then $2k - 7 = 3$ and $k = 5$ and $3^m = 6$, which is not possible.
- If $n = 2$, then $2k - 7 = 9$ and $k = 8$ and $3^m = 27$, which means $m = 3$.

Thus, $k = 8$ is another solution. Therefore, $k = 2, 3, 4$, and 8 are the solutions. ■

3. Define the sequence $\{a_i\}$ by $a_0 = 1$, $a_1 = 4$, and $a_{n+1} = 5a_n - a_{n-1}$ for all $n \geq 1$. Show that all terms of the sequence are of the form $c^2 + 3d^2$ for some integers c and d.

Solution: The first few terms are $1 = 1^2 + 3(0)^2$, $4 = 1^2 + 3(1)^2$, $19 = 4^2 + 3(1)^2$, $91 = 4^2+3(1+4)^2$, $436 = 19^2+3(1+4)^2$, $2089 = 19^2+3(1+4+19)^2$, $9573 = 91^2+3(1+4+19)^2$.

We claim that for all $n \geq 1$, $a_{2n} = a_n^2 + 3(a_0 + a_1 + \cdots + a_{n-1})^2$ and $a_{2n+1} = a_n^2 + 3(a_0 + a_1 + \cdots + a_n)^2$. Defining $b_n = a_0 + \cdots + a_n$, we restate these as $a_{2n} = a_n^2 + 3b_{n-1}^2$ and $a_{2n+1} = a_n^2 + 3b_n^2$.

Note that for all $n \in \mathbb{N}$, $b_n = b_{n-1} + a_n$.

We have verified these for small values. Suppose $k \geq 1$ and $a_{2k} = a_k^2 + 3b_{k-1}^2$ and $a_{2k+1} = a_k^2 + 3b_k^2$.

We first prove by induction that $a_n - a_{n-1} = 3b_{n-1}$ for all $n \geq 1$. This is true for $n = 1$. Suppose it is true for $n = k$. Then,

$$a_{k+1} - a_k = 5a_k - a_{k-1} - a_k = 3a_k + (a_k - a_{k-1}) = 3a_k + 3b_{k-1} = 3b_k.$$

Thus, it is indeed true for all n.

Then,

$$\begin{aligned}
a_{2k+2} &= 5a_{2k+1} - a_{2k} \\
&= 5a_k^2 + 15b_k^2 - a_k^2 - 3b_{k-1}^2 = 4a_k^2 + 12b_k^2 - 3b_{k-1}^2 + 3b_k^2 \\
&= 4a_k^2 + 12(a_k + b_{k-1})^2 - 3b_{k-1}^2 + 3b_k^2 \\
&= (4a_k - 3b_{k-1})^2 + 3b_k^2 \\
&= (a_k + 3b_{k-1})^2 + 3b_k^2 = a_{k+1}^2 + 3b_k^2
\end{aligned}$$

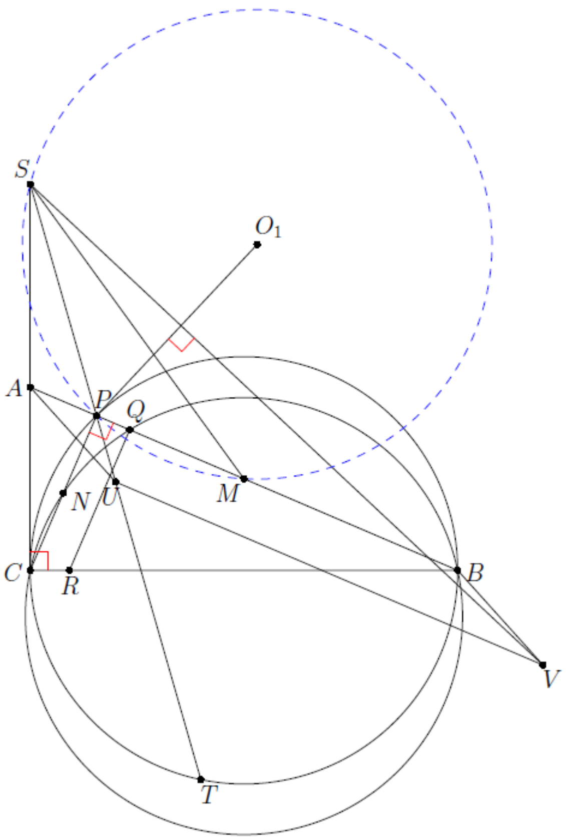
Therefore,

$$\begin{aligned}
a_{2k+3} &= 5a_{2k+2} - a_{2k+1} \\
&= 5a_{k+1}^2 + 15b_k^2 - a_k^2 - 3b_k^2 \\
&= 5a_{k+1}^2 + 9b_k^2 - a_k^2 + 3b_k^2 \\
&= 2a_{k+1}^2 + 9b_k^2 - a_k^2 - 6a_{k+1}b_k + 3a_{k+1}^2 + 6a_{k+1}b_k + 3b_k^2 \\
&= 2a_{k+1}^2 + 9b_k^2 - a_k^2 - 6a_{k+1}b_k + 3(a_{k+1} + b_k)^2 \\
&= a_{k+1}^2 + a_{k+1}^2 - 6a_{k+1}b_k + 9b_k^2 - a_k^2 + 3b_{k+1}^2 \\
&= a_{k+1}^2 + (a_{k+1} - 3b_k)^2 - a_k^2 + 3b_{k+1}^2 \\
&= a_{k+1}^2 + a_k^2 - a_k^2 + 3b_{k+1}^2 = a_{k+1}^2 + 3b_{k+1}^2
\end{aligned}$$

This completes our proof. ■

4. In acute triangle ABC with $\angle BAC > \angle BCA$, let P be the point on side BC such that $\angle PAB = \angle BCA$. The circumcircle of triangle APB meets side AC again at Q. Point D lies on segment AP such that $\angle QDC = \angle CAP$. Point E lies on line BD such that $CE = CD$. The circumcircle of triangle CQE meets segment CD again at F, and line QF meets side BC at G. Show that B, D, F, and G are concyclic.

Solution. Refer to the figure shown below.



Since $ABPQ$ is cyclic, we have $CP \cdot CB = CQ \cdot AC$. Also, we have $\triangle CAD \sim \triangle CDQ$, so $CD^2 = CQ \cdot AC$. This means that $CE^2 = CD^2 = CQ \cdot AC = CP \cdot CB$, so $\triangle CDP \sim \triangle CBD$ and $\triangle CEQ \sim \triangle CAE$. Thus, $\angle CBD = \angle CDP$ and, since $QECF$ is cyclic, $\angle CAE = \angle CEQ = \angle QFD$. Now, we see that

$$\begin{aligned}\angle EDC = \angle CBD + \angle DCB &= \angle CBD + \angle ACB - \angle ACD \\ &= \angle CBD + \angle ACB - (\angle CDP - \angle DAC) \\ &= \angle BAP + \angle DAC = \angle BAC\end{aligned}$$

and since triangle DCE is isosceles with $CD = CE$, we get $\angle DEC = \angle BAC$. It follows that $BAEC$ is cyclic, so $\angle GBD = \angle CBD = \angle CAE$. But $\angle CAE = \angle QFD$, so $\angle GBD = \angle QFD$ and therefore, $BDFG$ is cyclic. The desired conclusion follows. ■