22nd Philippine Mathematical Olympiad
18 January 2020
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$\boxed{\text{EASY}}$ 20 seconds, 2 points

1. Find the sum of the squares of the real roots of the equation $2x^4 - 3x^3 + 7x^2 - 9x + 3 = 0$.

   Answer: $\dfrac{5}{4}$

   Solution: By synthetic division by $(x - 1)$, we get

$$\begin{aligned} 2x^4 - 3x^3 + 7x^2 - 9x + 3 &= (x - 1)(2x^3 - x^2 + 6x - 3) \\ &= (x - 1)(2x - 1)(x^2 + 3). \end{aligned}$$

   Hence, the only real roots of $2x^4 - 3x^3 + 7x^2 - 9x + 3 = 0$ are 1 and $\frac{1}{2}$. and the sum is $1 + \frac{1}{4} = \frac{5}{4}$.

2. What is the remainder when 3^{2020} is divided by 73?

   Answer: 8

   Solution: By Fermat's Little Theorem, $3^{2016} = (3^{72})^{28} \equiv 1 (\text{mod } 73)$. Therefore, $3^{2020} \equiv 3^4 \equiv 8 (\text{mod } 73)$.

3. What is the largest integer k such that $k + 1$ divides

$$k^{2020} + 2k^{2019} + 3k^{2018} + \cdots + 2020k + 2021 \quad ?$$

   Answer: 1010

   Solution: The remainder when the polynomial $x^{2020} + 2x^{2019} + 3x^{2018} + \cdots + 2020x + 2021$ is divided by $x + 1$ is

$$(-1)^{2020} + 2(-1)^{2019} + 3(-1)^{2018} + \cdots + 2020(-1) + 2021 = 1010(-1) + 2021 = 1011.$$

   Therefore, $k+1$ divides $k^{2020} + 2k^{2019} + 3k^{2018} + \cdots + 2020k + 2021$ precisely when $k + 1$ divides 1011. The largest k for which this is true is 1010.

4. A right triangle has legs of lengths 3 and 4. Find the volume of the solid formed by revolving the triangle about its hypotenuse.

   Answer: $\dfrac{48\pi}{5}$ (cubic units)

   Solution: The solid consists of two conical solids with a common circular base of radius $\dfrac{3 \cdot 4}{5} = \dfrac{12}{5}$. If h_1 and h_2 are the heights of the two cones, then $h_1 + h_2 = 5$. Hence, the volume of the solid is

$$\frac{\pi}{3} \cdot \left(\frac{12}{5}\right)^2 \cdot (h_1 + h_2) = \frac{48\pi}{5}.$$

5. Suppose f is a second-degree polynomial for which $f(2) = 1$, $f(4) = 2$, and $f(8) = 3$. Find the sum of the roots of f.

   Answer: 18

   Solution: Let $f(x) = ax^2 + bx + c$. By substituting $x = 2, 4, 8$ we get the system of linear equations

$$\begin{aligned} 4a + 2b + c &= 1 \\ 16a + 4b + c &= 2 \\ 64a + 8b + c &= 3. \end{aligned}$$

   Solving this system of equations gives us $a = -\frac{1}{24}$, $b = \frac{3}{4}$, $c = -\frac{1}{3}$. By using Vieta's identities, the sum of roots is $-\frac{b}{a} = \frac{3}{4} \cdot 24 = 18$.

6. A triangle has side lengths 7, 11, 14. Find the length of its inradius.

Answer: $\dfrac{3\sqrt{10}}{4}$

Solution: We use Heron's formula. The semiperimeter of the triangle is 16. Thus, the area of the triangle is $\sqrt{(16)(16-7)(16-11)(16-2)} = 12\sqrt{10}$. Since the area of a triangle is the product of the semiperimeter and the inradius, the length of the inradius is $\boxed{\dfrac{3\sqrt{10}}{4}}$.

7. Suppose that $(1+\sec\theta)(1+\csc\theta) = 6$. Determine the value of $(1+\tan\theta)(1+\cot\theta)$.

Answer: $\dfrac{49}{12}$

Solution: The equation is equivalent to $1+\sin\theta+\cos\theta = 5\sin\theta\cos\theta$. Let $A := \sin\theta+\cos\theta$ and $B := \sin\theta\cos\theta$. Then $1+A = 5B$ and $A^2 = 1+2B$. As $1+A \neq 0$,

$$1+A = \frac{5(A^2-1)}{2} \implies 1 = \frac{5(A-1)}{2},$$

which gives $A = \frac{7}{5}$. Consequently, we get $B = \frac{12}{25}$. Hence,

$$(1+\tan\theta)(1+\cot\theta) = \frac{A^2}{B} = \frac{49}{25}\cdot\frac{25}{12} = \frac{49}{12}.$$

8. Determine the number of ordered quadruples (a, b, c, d) of positive integers such that $abcd = 216$.

Answer: 400

Solution: Since $216 = 2^3 3^3$, any positive divisor of 216 must be of the form $2^x 3^y$ for some integers x and y with $0 \le x, y \le 3$. Thus, we set $a = 2^{x_1}3^{y_1}, b = 2^{x_2}3^{y_2}, c = 2^{x_3}3^{y_3}$ and $d = 2^{x_4}3^{y_4}$, where $0 \le x_i, y_i \le 3$ are integers for $i = 1, \dots, 4$. We compute

$$2^3 3^3 = 216 = abcd = 2^{x_1+x_2+x_3+x_4}3^{y_1+y_2+y_3+y_4}$$

so the number of such ordered quadruples (a, b, c, d) is the number of ordered 8-tuples $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ of nonnegative integers such that $x_1+x_2+x_3+x_4 = y_1+y_2+y_3+y_4 = 3$. By stars-and-bars, this number is $\binom{3+4-1}{4-1}^2 = 20^2 = 400$.

9. A 10×1 rectangular pavement is to be covered by tiles which are either green or yellow, each of width 1 and of varying integer lengths from 1 to 10. Suppose you have an unlimited supply of tiles for each color and for each of the varying lengths. How many distinct tilings of the rectangle are there, if at least one green and one yellow tile should be used, and adjacent tiles should have different colors?

Answer: 1022

Solution: Note that the pavement is fixed and cannot be rotated, therefore a tiling is considered distinct from the reverse tiling. Also, note that the restriction that no two consecutive tiles can be of the same color can be addressed simply by treating consecutive tiles of the same color as one tile. Hence, the problem is just asking for the number of possible tilings that alternate both colors. For any division of the board into tiles, there are precisely two ways to color the tiles, as the coloring is determined solely by the color of the first tile. Now, to count the uncolored tilings, we divide the board into 10 squares using 9 dividers, and just count the number of subsets of the 9 dividers with at least one element. There are exactly $2^9 - 1 = 511$ such subsets, and thus $511 \cdot 2 = 1022$ such colorings.

10. Let $P = (3^1+1)(3^2+1)(3^3+1)\dots(3^{2020}+1)$. Find the largest value of the integer n such that 2^n divides P.

Answer: 3030

Solution: If k is even, then note that $3^k+1 \equiv 2 \pmod 4$ and so $2 \| 3^k+1$, i.e., $4 \nmid 3^k+1$. On the other hand, if k is odd, note that $3^k+1 \equiv 4 \pmod 8$ so $4 \| 3^k+1$, i.e., $4 \mid 3^k+1$ but $8 \nmid 3^k+1$. Thus the greatest value of m for which 2^m divides 3^k+1 is 2 if k is odd, and 1 if k is even. Summing up over $1 \le k \le 2020$ gives us $2+1+\dots+2+1 = 1010(2+1) = 3030$.

11. An infinite geometric series has sum 2020. If the first term, the third term, and the fourth term form an arithmetic sequence, find the first term.

    Answer: $1010(1+\sqrt{5})$

    Solution: Let a be the first term and r be the common ratio. Thus, $\frac{a}{1-r} = 2020$, or $a = 2020(1-r)$. We also have $ar^2 - a = ar^3 - ar^2$. Since the sum of the geometric series is nonzero, $a \neq 0$, and so we have $r^2 - 1 = r^3 - r^2$, or $r^3 - 2r^2 + 1 = 0$. Since the sum of the geometric series is finite, r cannot be 1, so $r^3 - 2r^2 + 1 = (r-1)(r^2 - r - 1) = 0$ implies $r^2 - r - 1 = 0$. Solving this equation gives $r = \frac{1-\sqrt{5}}{2}$ (since $|r| < 1$). This gives us $a = 2020\left(\frac{1+\sqrt{5}}{2}\right) = 1010(1+\sqrt{5})$.

12. Find the 2020th term of the following sequence:

$$1, 1, 3, 1, 3, 5, 1, 3, 5, 7, 1, 3, 5, 7, 9, 1, 3, 5, 7, 9, 11, \ldots$$

    Answer: 7

    Solution: We have that for each $n \in \mathbb{N}$, the $(1 + 2 + \cdots + n)$th term is $2n - 1$. The first $n + 1$ odd positive integers are then listed. Observe that the largest triangular number less than or equal to 2020 is $\frac{63(64)}{2} = 2016$. Therefore, the 2020th term is 7.

13. How many infinite arithmetic sequences of positive integers are there which contain the numbers 3 and 39?

    Answer: 12

    Solution: The common difference should be a positive divisor of $39 - 3 = 36 = 2^2 \cdot 3^2$, which has $(2+1)(2+1)$ factors. If the common difference is 1, then any one of 1, 2, or 3 may be the first term. If the common difference is 2, then the first term may be either 1 or 3 only. For the other 7 possible common differences, the first term must be 3. Thus, there are 12 such arithmetic progressions in all.

14. What is the sum of all four-digit numbers that each use the digits 2, 4, 6, and 8 exactly once?

    Answer: 133,320

    Solution: Every digit appears in each decimal place exactly 6 times. Therefore, the sum is $6 \cdot (2 + 4 + 6 + 8) \cdot 1111 = 133{,}320$.

15. One of the biggest mathematical breakthroughs in 2019 was progress on an 82-year old problem by the renowned mathematician and Fields medalist Terence Tao.

    Consider the function

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

    Starting with any positive integer n, it was conjectured that recursive applications of the above function always lead to 1.

    While a general proof of this result still eludes the mathematical community, Tao was able to show that if there are counterexamples to this conjecture, their frequency approaches 0 as n increases. What is the surname of the German mathematician who proposed this conjecture in 1937?

    Answer: Collatz

$\boxed{\text{AVERAGE}}$ 45 seconds, 3 points

1. What is the probability that a rectangle with perimeter 36 cm has area greater than 36 cm^2?

   Answer: $\dfrac{\sqrt{5}}{3}$

   Solution: Let x and y be the lengths of the sides of the rectangle. We are looking for the probability that $xy > 36$ given that $2x+2y = 36$. Equivalently, we compute the probability that $x(18-x) > 36$ given $0 < x < 18$.

   Now, $x(18-x) > 36 \Leftrightarrow x^2 - 18x + 36 < 0 \Leftrightarrow (x-9)^2 < 45 \Leftrightarrow 9 - 3\sqrt{5} < x < 9 + 3\sqrt{5}$, where $9 - 3\sqrt{5} > 0$. Thus, the probability we are computing is $\dfrac{(9+3\sqrt{5}) - (9-3\sqrt{5})}{18-0} = \dfrac{\sqrt{5}}{3}$.

2. Let a and b be real numbers that satisfy the equations

$$\frac{a}{b} + \frac{b}{a} = \frac{5}{2} \quad \text{and} \quad a - b = \frac{3}{2}.$$

   Find all possible values of $a^2 + 2ab + b^2 + 2a^2b + 2ab^2 + a^2b^2$.

   Answer: 0 and 81

   Solution: From the 2nd equation, we have $a^2+b^2 = \frac{9}{4}+2ab$. Using this for the 1st equation, we have $\frac{a^2+b^2}{ab} = \frac{\frac{9}{4}+2ab}{ab} = \frac{5}{2}$. and it can be solved that $ab = \frac{9}{2}$. Moreover, from the 2nd equation, we have $(a+b)^2 = \frac{9}{4}+4ab = \frac{81}{4}$. Thus, $a+b = \frac{9}{2}$ or $-\frac{9}{2}$ and $a^2+2ab+b^2+2a^2b+2ab^2+a^2b^2 = (ab+a+b)^2 = 81$ or 0.

3. How many permutations of the string "000011112222" contain the substring "2020"?

   Answer: 3575

   Solution: Removing the string "2020", there are two 0's, four 1's, and two 2's remaining. There are $\frac{8!}{2!4!2!} = 420$ ways to arrange these digits, multiplied to 9 possible placements for the string "2020", for a product of 3780.

   However, by PIE, we still need to subtract the number of arrangements with string "202020" and add back the number of arrangements with string "20202020", each of which account for $7 \cdot \frac{6!}{1!4!1!} = 210$ and $5 \cdot \frac{4!}{0!4!0!} = 5$ arrangements, respectively. This gives us a total of $3780 - 210 + 5 = 3575$ possible permutations.

4. Kyle secretly selects a subset of $\{1,2,3,4\}$. Albert also secretly selects a subset of $\{1,2,3,4\}$. What is the probability that their chosen subsets have at least one element in common?

   Answer: $\dfrac{175}{256}$

   Solution: Let A and B be the subsets selected by Kyle and Albert, respectively. We first find the probability that two subsets A and B are disjoint. For each $k \in \{0,1,\ldots,4\}$, we choose an arbitrary subset A with k elements. In order for A and B to be disjoint, B must be a subset of the complement $\{1,2,3,4\} \setminus A$ with $4-k$ elements. Thus, for each $k \in \{0,1,\ldots,4\}$, there are $\binom{4}{k}$ subsets with k elements and fixing one of such subsets (say A), there are 2^{4-k} choices for B (note that 2^{4-k} is the number of subsets of $\{1,2,3,4\} \setminus A$). We see that the number of ordered pairs (A,B) of subsets with $A \cap B = \varnothing$ is

$$\sum_{k=0}^{4} \binom{4}{k} 2^{4-k} = 16 \sum_{k=0}^{4} \binom{4}{k} \left(\frac{1}{2}\right)^k = 16\left(1+\frac{1}{2}\right)^4 = 81.$$

   As there are $2^4 = 16$ subsets of $\{1,2,3,4\}$, there are $16^2 = 256$ possible ordered pairs of subsets. Hence, the probability that two subsets A and B are disjoint is $\frac{81}{256}$ and the probability that A and B have at least one element in common is $1 - \frac{81}{256} = \frac{175}{256}$.

5. A 20×19 rectangle is plotted on the Cartesian plane with one corner at the origin and with sides parallel to the coordinate axes. How many unit squares do the two diagonals of this rectangle pass through?

   Answer: 74

   Solution: Suppose that one corner of the rectangle is on $(20, 19)$. First of all, note that 20 and 19 are relatively prime. This means that the line does not intersect any vertex of a unit square in the interior of the grid.

   Now, consider the diagonal from (0,0) to (20,19). This diagonal intersects exactly 19 vertical segments and 18 horizontal segments. Here, each intersection point represents an entry point into a new unit square. Including the original unit square (which has the origin as a corner), this accounts for a total of $19 + 18 + 1 = 38$ unit squares.

   Similarly, the other diagonal of the rectangle passes through 38 unit squares. Finally, since the diagonals intersect at the center, the center must have been an entry point for one diagonal and an exit point for the other, and vice versa. These pertain to the two unit squares that both diagonals pass through.

   Therefore, the diagonals pass through a total of $2(38) - 2 = 74$ unit squares.

6. In $\triangle ABC$, $AB = 20$, $BC = 21$, and $CA = 29$. Point M is on side AB with $\dfrac{AM}{MB} = \dfrac{3}{2}$, while point N is on side BC with $\dfrac{CN}{NB} = 2$. P and Q are points on side AC such that the line MP is parallel to BC and the line NQ is parallel to AB. Suppose that MP and NQ intersect at point R. Find the area of $\triangle PQR$.

   Answer: $\dfrac{224}{15}$

   Solution: We use similar triangles here. Note that triangles ABC, AMP, QNC and QRP are all similar right (by the Pythagorean theorem, since $20^2 + 21^2 = 29^2$) triangles by AA similarity and corresponding angle theorem. We see that $AP = \frac{3}{5} \cdot 29 = \frac{87}{5}$ and $CQ = \frac{2}{3} \cdot 29 = \frac{58}{3}$.

   Hence, $PQ = AP + QC - AC = \frac{87}{5} + \frac{58}{3} - 29 = \frac{116}{15} = 29(\frac{4}{15})$. Thus, the ratio of similitude between QRP and ABC is $\frac{4}{15}$, and the area of triangle QRP is $(\frac{16}{225})(\frac{1}{2})(20)(21) = \frac{224}{15}$.

7. In a race with six runners, A finished between B and C, B finished between C and D, and D finished between E and F. If each sequence of winners in the race is equally likely to occur, what is the probability that F placed last?

   Answer: $\dfrac{5}{16}$

   Solution: Suppose $A > B$ means that A finished earlier than B. Then, the criteria imply the following:

   (a) either $B > A > C$ or $C > A > B$
   (b) either $E > D > F$ or $F > D > E$
   (c) either $D > B > C$ or $C > B > D$

   Combining Criteria 1 and 2 together implies either $D > B > A > C$ or $D < B < A < C$. When $D > B > A > C$, there's one place before D, and four after D (between the letters). Furthermore, there are two ways to permute E and F. Thus, there are $(1)(4)(2) = 8$ possible sequences, among which only 1 has F being last. On the other hand, when $C > A > B > D$, there are again 8 possibilities. If F is last, E still has 4 possible places before D.

   Thus, the required probability is $\dfrac{1+4}{8+8} = \dfrac{5}{16}$.

8. Suppose $A = \{1, 2, ..., 20\}$. Call B a *visionary* set of A if $B \subseteq A$, B contains at least one even integer, and $|B| \in B$, where $|B|$ is the cardinality of set B. How many *visionary* sets does A have?

   Answer: $2^{19} - 2^8$ or $2^{19} - 256$

   Solution: Ignoring the constraint that all visionary sets must contain at least one even integer, we count a total of

$$\sum_{i=0}^{19} \binom{19}{i} = 2^{19}$$

   sets. From this total, we subtract the number of sets that only contain odd numbers. Evidently, $|B|$ has to be odd. Since there are only 10 odd numbers in A, $|B|$ can only be 1, 3, 5, 7, or 9. The total number of instances when this happens is

$$\sum_{i=0}^{4} \binom{9}{2i} = \frac{1}{2}\sum_{i=0}^{9} \binom{9}{i} = \frac{1}{2}(2^9) = 2^8$$

   sets. Therefore, there are a total of $2^{19} - 2^8$ visionary sets.

9. Given triangle ABC, let D be a point on side AB and E be a point on side AC. Let F be the intersection of BE and CD. If $\triangle DBF$ has an area of 4, $\triangle BFC$ has an area of 6, and $\triangle FCE$ has an area of 5, find the area of quadrilateral $ADFE$.

   Answer: $\dfrac{105}{4}$ or 26.25

   Solution: Let the area of quadrilateral $ADFE$ be x. By Menelaus' Theorem, $\dfrac{AD}{DB} \cdot \dfrac{BF}{FE} \cdot \dfrac{EC}{CA} = 1$. Since $\dfrac{AD}{DB} = \dfrac{x+5}{10}$, $\dfrac{BF}{FE} = \dfrac{6}{5}$, and $\dfrac{EC}{CA} = \dfrac{11}{x+15}$, we have $\dfrac{66(x+5)}{50(x+15)} = 1$, or $x = \dfrac{105}{4}$ or 26.25.

10. If $a^3 + b^3 + c^3 = 3abc = 6$ and $a^2 + b^2 + c^2 = 8$, find the value of

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

   Answer: -8

   Solution: Since a, b, c are distinct and $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$, then $a+b+c = 0$. From $a+b+c = 0$ and $a^2+b^2+c^2 = 8$, we have $ab+bc+ca = -4$ and $a^2b^2 + b^2c^2 + c^2a^2 = (ab+bc+ca)^2 - 2abc(a+b+c) = (ab+bc+ca)^2 = 16$. Thus,

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} = \frac{ab}{-c} + \frac{bc}{-a} + \frac{ca}{-b} = \frac{-(a^2b^2 + b^2c^2 + c^2a^2)}{abc} = -\frac{16}{2} = -8$$

DIFFICULT 90 seconds, 6 points

1. Compute the sum of all possible distinct values of $m + n$ if m and n are positive integers such that

$$\operatorname{lcm}(m, n) + \gcd(m, n) = 2(m+n) + 11.$$

   Answer: 32

   Solution: Let $d = \gcd(m, n)$. We consider the following cases:

   (a) Suppose $d = 1$. Then the equation becomes $mn+1 = 2m+2n+11$ or $(m-2)(n-2) = 14$. As m and n are coprime in this case, exactly one of them is even. This implies that exactly one of the factors $m-2$ and $n-2$ of 14 is even. Thus, we have $(m-2, n-2) \in \{(14,1), (7,2), (2,7), (1,14)\}$ so that $(m, n) \in \{(16,3), (9,4), (4,9), (3,16)\}$. The sum of all possible values of $m+n$ is $19 + 13 = 32$.

(b) Suppose $d \geq 2$. As $d \mid \operatorname{lcm}(m, n)$, we see that d divides $2(m+n)+11$. But d divides both m and n, so $d \mid 11$. This forces $d = 11$. Plugging $m = 11x$ and $n = 11y$ into the given equation, where $\gcd(x, y) = 1$, we have $\operatorname{lcm}(11x, 11y) + \gcd(11x, 11y) = 2(11x + 11y) + 11$, which is equivalent to

$$11\operatorname{lcm}(x, y) + 11 = 2(11x + 11y) + 11 \implies xy = 2(x + y) \implies (x-2)(y-2) = 4.$$

Exactly one of the factors $x-2$ and $y-2$ of 4 is even. We then get $(x-2, y-2) \in \{(4,1), (1,4)\}$ and $(x, y) \in \{(6,3), (3,6)\}$, contradicting $\gcd(x, y) = 1$.

Hence, the sum of all possible values of $m+n$ satisfying $\operatorname{lcm}(m, n) + \gcd(m, n) = 2(m+n) + 11$ is 32.

2. In convex pentagon $ABCDE$, $AB = BC$, $CD = DE$, $\angle ABC = 100^\circ$, $\angle CDE = 80^\circ$, and $BD^2 = \dfrac{100}{\sin 100^\circ}$. Find the area of the pentagon.

Answer: 50

Solution: Let $AB = BC = p$, $AC = r$, $CD = DE = q$, $CE = s$, and $\theta = \angle ACE$. Then $r = 2p\cos 40^\circ$, $s = 2q\cos 50^\circ$, and

$$\begin{aligned} \frac{100}{\sin 100^\circ} &= BD^2 = p^2 + q^2 - 2pq\cos(90^\circ + \theta) \\ &= p^2 + q^2 + 2pq\sin\theta \end{aligned}$$

Thus, the required area is given by

$$\begin{aligned} \frac{1}{2}p^2\sin 100^\circ + \frac{1}{2}q^2\sin 80^\circ + \frac{1}{2}rs\sin\theta &= \frac{1}{2}p^2\sin 100^\circ + \frac{1}{2}q^2\sin 80^\circ + \frac{1}{2}(4pq\cos 40^\circ\sin 40^\circ)\sin\theta \\ &= \frac{1}{2}p^2\sin 100^\circ + \frac{1}{2}q^2\sin 100^\circ + pq\sin 100^\circ\sin\theta \\ &= \frac{1}{2}\sin 100^\circ(p^2 + q^2 + 2pq\sin\theta) \\ &= \frac{1}{2}\sin 100^\circ\left(\frac{100}{\sin 100^\circ}\right) \\ &= 50 \end{aligned}$$

3. Consider an equilateral triangle with side 700. Suppose that one move consists of changing the length of any of the sides of a triangle such that the result will still be a triangle. Find the minimum number of moves to change the given triangle to an equilateral triangle with side 2.

Answer: 14

Solution: Work backwards by starting from $(2, 2, 2)$. The fastest way to do this is to lengthen the shortest side, making it as long as the sum of the other two sides. Denote k^- to be a real number arbitrarily close to k, but less than k.

First move: $(2, 2, 4^-)$, Second move: $(2, 4^-, 6^-)$, Third move: $(4^-, 6^-, 10^-)$, and so on. Notice that it follows double of the Fibonacci sequence.

$$2, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, 288, 466$$

The side lengths after the 11th move are $(178^-, 288^-, 466^-)$. After 3 moves, we can change all the sides to 700 as follows:

Step 12: $(288, 466, 700)$

Step 13: $(466, 700, 700)$

Step 14: $(700, 700, 700)$

Thus, the minimum number of moves is 14.

4. A fixed point of a function f is a value of x for which $f(x) = x$. Let f be the quadratic function defined by $f(x) = x^2 - cx + c$ where $c \in \mathbb{R}$. Find, in interval notation, the set consisting of all values of c for which $f \circ f$ has four distinct fixed points.

   Answer: $(-\infty, -1) \cup (3, +\infty)$

   Solution: First, observe that both $x = c$ and $x = 1$ are fixed points of f, and thus fixed points of $f \circ f$. Indeed, a fixed point of f is a value of x such that $f(x) - x = 0$, and we have $f(x) - x = x^2 - (c+1)x + c = (x-1)(x-c)$. Moreover, $(f \circ f)(x) - x$ is a quartic polynomial in x, and its roots are the fixed points of $f \circ f$. Thus, it follows that $(f \circ f)(x) - x = (x-1)(x-c)g(x)$ for some quadratic polynomial g. Explicitly solving gives us $(f \circ f)(x) - x = (x-c)(x-1)(x^2 - (c-1)x + 1)$.

   Thus, we need only $x^2 - (c-1)x + 1$ to have two distinct roots, neither of which is equal to c or 1. This means that $(c-1)^2 - 4 > 0$, i.e. $(c+1)(c-3) > 0$. Thus, we want $c < -1$ or $c > 3$. We check now that 1 and c are not roots: we get if $x = 1$, $x^2 - (c-1)x + 1 = 3 - c \neq 0$ (since $c \neq 3$) and if $x = c$, $x^2 - (c-1)x + 1 = c + 1 \neq 0$ since $c \neq -1$.

   Thus, all $c \in (-\infty, -1) \cup (3, +\infty)$ work.

5. For a positive integer n, denote by $\varphi(n)$ the number of positive integers $k \leq n$ relatively prime to n. How many positive integers n less than or equal to 100 are divisible by $\varphi(n)$?

   Answer: 16

   Solution: We claim that any such integer n must be either equal to 1 or of the form $2^a 3^b$, where $a \geq 1$ and $b \geq 0$.

   First, we note that if $n > 1$, it must be even. This is because if n admits a prime factorization $n = \prod_{i=1}^{k} p_i^{r_i}$ over distinct primes p_i, then $\varphi(n) = \prod_{i=1}^{k} (p_i - 1) p_i^{r_i - 1}$l in particular, if an odd prime p_i divides n then $p_i - 1$ divides $\varphi(n)$, and consequently $\varphi(n)$ is even.

   Then, we note that at most one odd prime divides n. Suppose otherwise, i.e., $2^a p_1^{r_1} p_2^{r_2}$ is a part of the prime factorization of n for odd primes p_1, p_2. Then by the multiplicativity of φ, we know that $\varphi(2^a p_1^{r_1} p_2^{r_2}) = 2^{a-1}(p_1 - 1)p_1^{r_1 - 1}(p_2 - 1)p_2^{r_2 - 1}$ must divide $\varphi(n)$. Note that the largest power of 2 dividing n is 2^a, but since $p_1 - 1$ and $p_2 - 1$ are both even, $2^(a-1)2^2 = 2^{a+1}$ divides $\varphi(n)$. This is a contradiction.

   Finally, we show that the said odd prime p dividing n must in fact be equal to 3. Indeed, if $n = 2^a p^r$ where a and r are positive integers, we have $\frac{n}{\varphi(n)} = \frac{2p}{p-1} = 2 + \frac{2}{p-1}$. For this to be an integer, $p = 3$.

   Hence, we count all such integers of the form $n = 2^a 3^r$ by summing over all possible values of r (because there are fewer):

   If $r = 0$, then $0 \leq a \leq 6$. (This is the only case in which $a = 0$ is allowed.)
   If $r = 1$, then $1 \leq a \leq 5$.
   If $r = 2$, then $1 \leq a \leq 3$.
   If $r = 3$, then $a = 1$.

   This gives us a total of $7 + 5 + 3 + 1 = 16$ values of n.