23rd Philippine Mathematical Olympiad

Qualifying Stage, 20 February 2021

PART I. Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

1. In a convex polygon, the number of diagonals is 23 times the number of its sides. How many sides does it have?

(a) 46 (b) 49 (c) 66 (d) 69

2. What is the smallest real number a for which the function $f(x) = 4x^2 - 12x - 5 + 2a$ will always be nonnegative for all real numbers x?

(a) 0 (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 7

3. In how many ways can the letters of the word *PANACEA* be arranged so that the three As are not all together?

(a) 540 (b) 576 (c) 600 (d) 720

4. How many ordered pairs of positive integers (x, y) satisfy $20x + 21y = 2021$?

(a) 4 (b) 5 (c) 6 (d) infinitely many

5. Find the sum of all k for which $x^5 + kx^4 - 6x^3 - 15x^2 - 8k^3x - 12k + 21$ leaves a remainder of 23 when divided by $x + k$.

(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{5}{8}$ (d) $\frac{3}{4}$

6. In rolling three fair twelve-sided dice simultaneously, what is the probability that the resulting numbers can be arranged to form a geometric sequence?

(a) $\frac{1}{72}$ (b) $\frac{5}{288}$ (c) $\frac{1}{48}$ (d) $\frac{7}{288}$

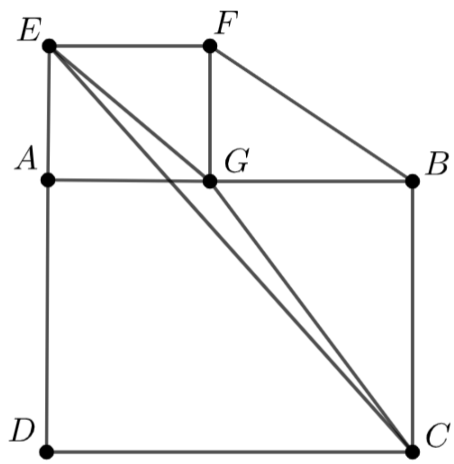
7. How many positive integers n are there such that $\dfrac{n}{120 - 2n}$ is a positive integer?

(a) 2 (b) 3 (c) 4 (d) 5

8. Three real numbers a_1, a_2, a_3 form an arithmetic sequence. After a_1 is increased by 1, the three numbers now form a geometric sequence. If a_1 is a positive integer, what is the smallest positive value of the common difference?

(a) 1 (b) $\sqrt{2} + 1$ (c) 3 (d) $\sqrt{5} + 2$

9. Point G lies on side AB of square $ABCD$ and square $AEFG$ is drawn outwards $ABCD$, as shown in the figure below. Suppose that the area of triangle EGC is $1/16$ of the area of pentagon $DEFBC$. What is the ratio of the areas of $AEFG$ and $ABCD$?



(a) $4 : 25$ (b) $9 : 49$ (c) $16 : 81$ (d) $25 : 121$

10. In how many ways can 2021 be written as a sum of two or more consecutive integers?

(a) 3 (b) 5 (c) 7 (d) 9

11. In quadrilateral $ABCD$, $\angle CBA = 90°$, $\angle BAD = 45°$, and $\angle ADC = 105°$. Suppose that $BC = 1 + \sqrt{2}$ and $AD = 2 + \sqrt{6}$. What is the length of AB?

(a) $2\sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $3 + \sqrt{2}$ (d) $3 + \sqrt{3}$

12. Alice tosses two biased coins, each of which has a probability p of obtaining a head, simultaneously and repeatedly until she gets two heads. Suppose that this happens on the rth toss for some integer $r \geq 1$. Given that there is 36% chance that r is even, what is the value of p?

(a) $\dfrac{\sqrt{7}}{4}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{\sqrt{2}}{2}$ (d) $\dfrac{3}{4}$

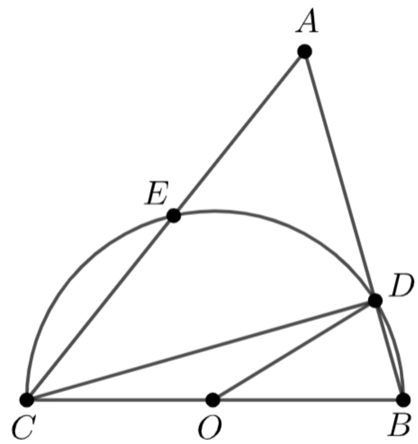
13. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t and $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t. How many real numbers x between 1 and 23 satisfy $\lfloor x \rfloor \{x\} = 2\sqrt{x}$?

(a) 18 (b) 19 (c) 20 (d) 21

14. Find the remainder when $\displaystyle\sum_{n=2}^{2021} n^n$ is divided by 5.

(a) 1 (b) 2 (c) 3 (d) 4

15. In the figure below, BC is the diameter of a semicircle centered at O, which intersects AB and AC at D and E respectively. Suppose that $AD = 9$, $DB = 4$, and $\angle ACD = \angle DOB$. Find the length of AE.



(a) $\frac{117}{16}$ (b) $\frac{39}{5}$ (c) $2\sqrt{13}$ (d) $3\sqrt{13}$

PART II. All answers are positive integers. Do not use commas if there are more than 3 digits, e.g., type 1234 instead of $1,234$. A positive fraction a/b is in lowest terms if a and b are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. Consider all real numbers c such that $|x - 8| + |4 - x^2| = c$ has exactly three real solutions. The sum of all such c can be expressed as a fraction a/b in lowest terms. What is $a + b$?

17. Find the smallest positive integer n for which there are exactly 2323 positive integers less than or equal to n that are divisible by 2 or 23, but not both.

18. Let $P(x)$ be a polynomial with integer coefficients such that $P(-4) = 5$ and $P(5) = -4$. What is the maximum possible remainder when $P(0)$ is divided by 60?

19. Let $\triangle ABC$ be an equilateral triangle with side length 16. Points D, E, F are on CA, AB, and BC, respectively, such that $DE \perp AE$, $DF \perp CF$, and $BD = 14$. The perimeter of $\triangle BEF$ can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

20. How many subsets of the set $\{1, 2, 3, \ldots, 9\}$ do not contain consecutive odd integers?

21. For a positive integer n, define $s(n)$ as the smallest positive integer t such that n is a factor of $t!$. Compute the number of positive integers n for which $s(n) = 13$.

22. Alice and Bob are playing a game with dice. They each roll a die six times, and take the sums of the outcomes of their own rolls. The player with the higher sum wins. If both players have the same sum, then nobody wins. Alice's first three rolls are 6, 5, and 6, while Bob's first three rolls are 2, 1, and 3. The probability that Bob wins can be written as a fraction a/b in lowest terms. What is $a + b$?

23. Let $\triangle ABC$ be an isosceles triangle with a right angle at A, and suppose that the diameter of its circumcircle Ω is 40. Let D and E be points on the arc BC not containing A such that D lies between B and E, and AD and AE trisect $\angle BAC$. Let I_1 and I_2 be the incenters of $\triangle ABE$ and $\triangle ACD$ respectively. The length of I_1I_2 can be expressed in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

24. Find the number of functions f from the set $S = \{0, 1, 2, \ldots, 2020\}$ to itself such that, for all $a, b, c \in S$, all three of the following conditions are satisfied:

(i) If $f(a) = a$, then $a = 0$;

(ii) If $f(a) = f(b)$, then $a = b$; and

(iii) If $c \equiv a + b \pmod{2021}$, then $f(c) \equiv f(a) + f(b) \pmod{2021}$.

25. A sequence $\{a_n\}$ of real numbers is defined by $a_1 = 1$ and for all integers $n \geq 1$,

$$a_{n+1} = \frac{a_n \sqrt{n^2 + n}}{\sqrt{n^2 + n + 2a_n^2}}.$$

Compute the sum of all positive integers $n < 1000$ for which a_n is a rational number.

Answers

Part I. (2 points each)

1. B

2. D

3. D

4. B

5. B

6. D

7. B

8. B

9. A

10. C

11. C

12. A

13. A

14. D

15. B

Part II. (5 points each)

16. 93

17. 4644

18. 41

19. 31

20. 208

21. 792

22. 3895

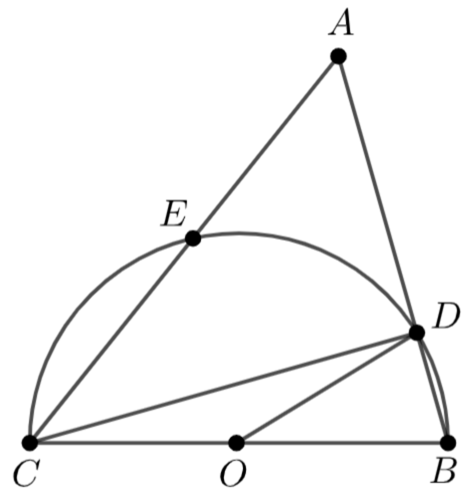
23. 20

24. 1845

25. 131

Solutions to selected problems:

15. In the figure below, BC is the diameter of a semicircle centered at O, which intersects AB and AC at D and E respectively. Suppose that $AD = 9$, $DB = 4$, and $\angle ACD = \angle DOB$. Find the length of AE.



Solution. Let $\angle DOB = \angle EOC = \alpha$. Note that $\angle DCB = \frac{\alpha}{2}$. Also, note that $\tan\frac{\alpha}{2} = \frac{4}{DC}$ and $\tan\alpha = \frac{9}{DC} = \frac{9}{4}\tan\frac{\alpha}{2}$. Let $x = \tan\frac{\alpha}{2}$. By the double-angle formula,

$$\begin{aligned}\frac{9}{4}x &= \frac{2x}{1-x^2}\\ \frac{9}{4}x - \frac{9}{4}x^3 &= 2x\\ \frac{1}{4}x\left(1-9x^2\right) &= 0\end{aligned}$$

and thus $x = 0$ or $x = \pm\frac{1}{3}$. Clearly, only $x = \frac{1}{3}$ is possible here. Thus, $CD = 12$. Note also that $\angle CDB = \angle ADC = 90^\circ$, and so by the Pythagorean theorem, $AC = \sqrt{9^2+12^2} = 15$.

Finally, by the power of a point theorem, we have $AE \cdot AC = AD \cdot AB$ and so $AE \cdot 15 = 9(9+4)$, which gives us $AE = \frac{117}{15} = \boxed{\dfrac{39}{5}}$.

17. Find the smallest positive integer n for which there are exactly 2323 positive integers less than or equal to n that are divisible by 2 or 23, but not both.

Solution. The number of positive integers from 1 to n that are divisible by 2 or 23, but not both, is

$$f(n) = \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{23}\right\rfloor - 2\left\lfloor\frac{n}{46}\right\rfloor.$$

We need to find $f(n) = 2323$, which can be done by some trial and error.

Note that if n is a multiple of 2 and 23, then the floor divisions are non-rounded exact divisions, in which case,

$$\frac{n}{2} + \frac{n}{23} - 2\,\frac{n}{46} = \frac{n}{2}.$$

So, for example, $f(4646) = 2323$. We can then just tick downwards.

Since 4646 and 4645 do not satisfy our criteria, we know that $f(4646) = f(4645) = f(4644)$, i.e. we can remove 4646 and 4645 and the count does not go down (we weren't counting them). However, 4644 *does* satisfy our criteria, so $f(4643) = 2322$.

Note that by definition, this function is non-decreasing. Thus, we conclude that $n = \boxed{4644}$ is the first n that satisfies the given conditions.

18. Let $P(x)$ be a polynomial with integral coefficients such that $P(-4) = 5$ and $P(5) = -4$. What is the maximum possible remainder when $P(0)$ is divided by 60?

Solution.

$$0 - (-4)|P(0) - P(-4) \text{ or } 4|P(0) - 5, \text{ so } P(0) \equiv 5 (\text{mod } 4)$$

$$5 - 0|P(5) - P(0) \text{ or } 5| - 4 - P(0), \text{ so } P(0) \equiv -4 (\text{mod } 5)$$

By the Chinese Remainder Theorem, there is a solution r that satisfies both of the previous equations, and this solution is unique modulo $4 \cdot 5 = 20$. It is easy to verify that this solution is 1. Thus, $P(0) \equiv 1 (\text{mod } 20)$. This implies that $P(0)$ can be $1, 21$, or $41 (\text{mod } 60)$. The largest remainder $\boxed{41}$ is indeed achievable, for example by the polynomial $-2(x + 4)(x - 5) + 1 - x$.

19. Let $\triangle ABC$ be an equilateral triangle with side length 16. Points D, E, F are on CA, AB, and BC, respectively, such that $DE \perp AE$, $DF \perp CF$, and $BD = 14$. The perimeter of $\triangle BEF$ can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

Solution. Let $\overline{AD} = 2x$, then $\overline{DC} = 16 - 2x$. Since $\triangle DAE$ and $\triangle DCF$ are both 30-60-90 triangles, then $\overline{AE} = \overline{AD}/2 = x$, $\overline{ED} = x\sqrt{3}$ and $\overline{CF} = \overline{DC}/2 = 8 - x$, $\overline{FD} = (8 - x)\sqrt{3}$. Since $\overline{AB} = \overline{BC} = 16$, then $\overline{EB} = 16 - x$ and $\overline{FB} = 8 + x$. Using Pythagorean Theorem on $\triangle DEB$, we have

$$\begin{aligned} \overline{ED}^2 + \overline{EB}^2 &= \overline{BD}^2 \\ (x\sqrt{3})^2 + (16 - x)^2 &= 14^2 \\ 3x^2 + 256 - 32x + x^2 &= 196 \\ x^2 - 8x + 15 &= 0 \\ (x - 5)(x - 3) &= 0 \\ x &= 5, 3 \end{aligned}$$

Choosing either of the two values of x will give the same result for the perimeter of $\triangle BEF$. Suppose we choose $x = 5$, then $\overline{EB} = 11$ and $\overline{FB} = 13$. By Cosine Law on $\triangle BEF$, we have

$$\begin{aligned} \overline{EF}^2 &= \overline{EB}^2 + \overline{FB}^2 - 2 \cdot \overline{EB} \cdot \overline{FB} \cos 60^\circ \\ \overline{EF}^2 &= 11^2 + 13^2 - 2(11)(13)(1/2) \\ \overline{EF} &= \sqrt{121 + 169 - 143} = 7\sqrt{3} \end{aligned}$$

Thus, the perimeter of $\triangle BEF = 24 + 7\sqrt{3}$, which means that $a + b + c + d = 24 + 0 + 7 + 0 = \boxed{31}$.

20. How many subsets of the set $\{1, 2, 3, \ldots, 9\}$ do not contain consecutive odd integers?

Solution. Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain consecutive odd integers. We work on the following cases depending on whether such a subset contains 1 or not:

- If such a subset does not contain 1, then its elements must consist of elements from $\{2, 3, \ldots, n\}$. The number of subsets with no element smaller than 3 is a_{n-2}; for each of these subsets, we include 2 as well, giving another a_{n-2} subsets. Thus, there are $2a_{n-2}$ subsets in this case.
- If such a subset contains 1, then it must not contain 3 and its elements must consist of elements from $\{1, 2, 4, 5, \ldots, n\}$. The number of subsets with no element smaller than 5 is a_{n-4}; for each of these subsets, we take the union with a non-empty subset of $\{2, 4\}$ as well, giving another $3a_{n-4}$ subsets. Thus, there are $4a_{n-4}$ subsets in this case.

We now obtain the recurrence formula $a_n = 2a_{n-2} + 4a_{n-4}$. With $a_1 = 2, a_2 = 4, a_3 = 6$ and $a_4 = 12$, routine computation gives $a_9 = \boxed{208}$.

21. For a positive integer n, define $s(n)$ as the smallest positive integer t such that n divides $t!$. Compute the number of positive integers n for which $s(n) = 13$.

Solution. For a positive integer k, consider the set $A(k) = \{n \in \mathbb{N} : s(n) = k\}$ and we wish to find $\#A(13)$. From the definition of $s(n)$, any element of $A(k)$ must divide $k!$ but not $(k-1)!$. Thus, any element of $A(k)$ must be a divisor of $k!$ that is not a divisor of $(k-1)!$. We see that $\#A(k) = \sigma_0(k!) - \sigma_0((k-1)!)$, where $\sigma_0(n)$ is the number of divisors of n, so that $\#A(13) = \sigma_0(13!) - \sigma_0(12!)$.

Note that for any prime p, the highest power of p that divides $k!$ is p^e, where $e = \sum_{j=1}^{\infty} \lfloor n/p^j \rfloor$. Using this formula, we determine the prime factorization of 13!: $13! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1$. As $13! = 13 \cdot 12!$, we get $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$. Hence, we obtain

$$\#A(13) = 11 \cdot 6 \cdot 3 \cdot 2^3 - 11 \cdot 6 \cdot 3 \cdot 2^2 = 198 \cdot 4 = \boxed{792}.$$

22. Alice and Bob are playing a game with dice. They each roll a die six times, and take the sums of the values of their own rolls. The player with the higher sum wins. If both players have the same sum, then nobody wins. Alice's first three rolls are 6, 5, and 6, while Bob's first three rolls are 2, 1, and 3. The probability that Bob wins can be written as a fraction a/b in lowest terms. What is $a + b$?

Solution. Let a_i denote the value of Alice's ith roll, and b_i denote the value of Bob's ith roll. For Bob to win, the following inequality must hold:

$$6 + 5 + 6 + a_4 + a_5 + a_6 < 2 + 1 + 3 + b_4 + b_5 + b_6$$

Rearranging yields

$$(a_4 - 1) + (a_5 - 1) + (a_6 - 1) + (6 - b_4) + (6 - b_5) + (6 - b_6) < 4.$$

Let $x_i = a_{i+3} - 1$ and $x_{i+3} = 6 - b_{i=3}$ for $i = 1, 2, 3$. The inequality then simplifies to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 4.$$

We claim that each nonnegative integer solutions to this inequality correspond to a valid situation of dice rolls. Note that the previous inequality implies that $0 \leq x_i < 4$, so $1 \leq x_i + 1 < 5$ and $2 < 6 - x_i \leq 6$. Thus, the corresponding dice rolls for both Alice and Bob are within the bounds.

The expression on the left can have a value of either $0, 1, 2$, or 3. Thus, the number of nonnegative integer solutions of the aforementioned equation is $\binom{5}{5} + \binom{6}{5} + \binom{7}{5} + \binom{8}{5} = 1 + 6 + 21 + 56 = 84$. Thus, the probability that Bob wins is $\dfrac{84}{6^6} = \dfrac{7}{3888}$, and so $a + b = 7 + 3888 = \boxed{3895}$.

23. Let $\triangle ABC$ be an isosceles triangle with a right angle at A, and suppose that the diameter of its circumcircle Ω is 40. Let D and E be points on the arc BC not containing A such that D lies between B and E, and AD and AE trisect $\angle BAC$. Let I_1 and I_2 be the incenters of $\triangle ABE$ and $\triangle ACD$ respectively. The length of I_1I_2 can be expressed in the form $a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}$, where a, b, c, and d are integers. Find $a+b+c+d$.

Solution. Let O be the center of Ω. Note that $\angle OBD = \angle CBD = \frac{\pi}{3}$, so $\triangle OBD$ is an equilateral triangle. Thus, $BD = BO = \frac{4}{2} = 2$. This implies that $EC = DE = BD = 2$. Clearly, I_1 and I_2 lie on the segments AD and AE respectively. It is well-known that $DI_1 = DB$, so $DI_1 = 20$. Similarly, $EI_2 = 20$. Now, note that $\angle EDI_1 = \angle EDC + \angle CDA = 30^\circ + 45^\circ = 75^\circ$. Similarly, $\angle DEI_2 = 75^\circ$. Looking at the isosceles trapezoid I_1I_2ED, the length of I_1I_2 must then be $DE - DI_1 \cos 75^\circ - EI_2 \cos 75^\circ = 2 - 40\cos 75^\circ = 20 - 40\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = 20 - \sqrt{6} + \sqrt{2}$, and so $a+b+c+d = 20+10+0-10 = \boxed{20}$.

24. How many functions f are there from the set $S = \{0, 1, 2, \ldots, 2020\}$ to itself such that, for all $a, b, c \in S$, all three of the following conditions are satisfied:

(i) If $f(a) = a$, then $a = 0$;

(ii) If $f(a) = f(b)$, then $a = b$; and

(iii) If $c \equiv a + b \pmod{2021}$, then $f(c) \equiv f(a) + f(b) \pmod{2021}$.

Solution. Note that, from (i), our function is completely determined by $f(1)$; i.e., $f(a) \equiv af(1) \pmod{2021}$. Then, from (i) and (ii), we need that $f(a) \neq 0$ if $a \neq 0$; otherwise, if $a \neq 0$ but $f(a) = 0$, $f(b) = f(a+b)$ for any b. Thus, if $a \neq 0$, we need that $af(1) \neq 0 \pmod{2021}$ for any $a \in S \setminus \{0\}$. If $\gcd(f(1), 2021) = d > 1$, then note that $a = \frac{2021}{d} \in S \setminus \{0\}$, and from (i), $f(a) \equiv af(1) \equiv 2021 \equiv 0 \pmod{2021}$. As we just established, this is not allowed. Hence, $\gcd(f(1), 2021) = 1$.

Moreover, from (iii), we need that $f(a) = af(1) \not\equiv a \pmod{2021}$ if $a \neq 0$; in other words, $2021 \nmid a(f(1)-1)$ if $a \neq 0$. By similar reasoning to earlier, suppose that $\gcd(f(1)-1, 2021) = d > 1$. Then $a = \frac{2021}{d} \in S \setminus \{0\}$, and $2021 \mid a(f(1)-1)$; thus, $f(a) = a$. We thus need that $\gcd(f(1)-1, 2021) = 1$ as well.

We count the number of integers that satisfy both these conditions. We use complementary counting here; thus, we start by counting those that fail to satisfy at least one condition. Indeed, we count the number of values of $f(1)$ that are *not* coprime to 2021. Either they are divisible by 43, or 47, or both. Since only 0 is divisible by both, by the principle of inclusion and exclusion, there are $47 + 43 - 1 = 89$ possible values of $f(1)$ so that $f(1)$ is not coprime to 2021. By the same reasoning, there are also 89 possible values of $f(1)$ such that $f(1) - 1$ is not coprime to 2021. Finally, we count the values of $f(1)$ for which neither $f(1)$ nor $f(1) - 1$ is coprime to 2021. This happens precisely when $43 \mid f(1)$ and $47 \mid f(1) - 1$, or $47 \mid f(1)$ and $43 \mid f(1) - 1$. By the Chinese remainder theorem, each of these possibilities gives one value of $f(1)$. Thus, by the principle of inclusion and exclusion, there are $2 \cdot 89 - 2 = 176$ such values of $f(1)$ that fail to satisfy at least one condition. This gives us $2021 - 176 = \boxed{1845}$ possible values of $f(1)$, and thus possible functions f.

25. A sequence $\{a_n\}$ of real numbers is defined by $a_1 = 1$ and $a_{n+1} = \dfrac{a_n\sqrt{n^2+n}}{\sqrt{n^2+n+2a_n^2}}$ for all integers $n \geq 1$. Compute the sum of all positive integers $n < 1000$ for which a_n is a rational number.

Solution. First, note that for $k \geq 1$,

$$a_{k+1}^2 = \frac{k(k+1)a_k^2}{k(k+1)+2a_k^2} \iff \frac{1}{a_{k+1}^2} - \frac{1}{a_k^2} = \frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1}$$

and summing the second equation from $k=1$ to $k=n-1$ with $n \geq 2$, we get

$$\frac{1}{a_n^2} - \frac{1}{a_1^2} = \sum_{k=1}^{n-1} \left(\frac{1}{a_{k+1}^2} - \frac{1}{a_k^2} \right) = \sum_{k=1}^{n-1} \left(\frac{2}{k} - \frac{2}{k+1} \right) = 2 - \frac{2}{n}.$$

Since $a_1 = 1$, we see that

$$\frac{1}{a_n^2} = 3 - \frac{2}{n} \iff a_n = \sqrt{\frac{n}{3n-2}}$$

for all integers $n \geq 1$ and we wish to find the sum of all positive integers $n < 1000$ such that $\frac{n}{3n-2}$ is a square of some rational number. To help us look for such integers n, we use the following lemma that provides integer solutions to the generalized Pell equation.

Lemma.[1] *Let d be a squarefree positive integer, and let a and b be positive integers such that $a^2 - db^2 = 1$. Set $u = a + b\sqrt{d}$. Then for each nonzero integer n, every solution of $x^2 - dy^2 = n$ is a power of u times $x + y\sqrt{d}$ where (x, y) is an integer solution of $x^2 - dy^2 = n$ with $|x| \leq \sqrt{|n|}(\sqrt{u}+1)/2$ and $|y| \leq \sqrt{|n|}(\sqrt{u}+1)/(2\sqrt{d})$.*

We now let $g = \gcd(n, 3n-2)$. Then $g \in \{1, 2\}$ since $g \mid 3n - (3n-2) = 2$. We now consider the following cases:

- Suppose $g = 1$. Then $n = y^2$ and $3n - 2 = x^2$ for some relatively prime positive integers x and y. This leads us to the generalized Pell equation $x^2 - 3y^2 = -2$. Set $u = 2 + \sqrt{3}$, with $(2, 1)$ being a solution of $x^2 - 3y^2 = 1$ in positive integers. We now look for the positive integer solutions (x, y) of $x^2 - 3y^2 = -2$ with $x \leq \sqrt{2}(\sqrt{u}+1)/2 \approx 2.07$ and $y \leq \sqrt{2}(\sqrt{u}+1)/(2\sqrt{3}) \approx 1.2$. We obtain $(x, y) = (1, 1)$ as the only such integer solution, so by the above lemma, we see that all positive integer solutions (x_k, y_k) of $x^2 - 3y^2 = -2$ are given by $x_k + y_k\sqrt{3} = (1+\sqrt{3})(2+\sqrt{3})^k$ for all integers $k \geq 0$. We now compute this product for small values of k:

k	0	1	2	3
$x_k + y_k\sqrt{3}$	$1+\sqrt{3}$	$5+3\sqrt{3}$	$19+11\sqrt{3}$	$71+41\sqrt{3}$

  As $n < 1000$, we require that $y_k \leq 31$, so the positive integer solutions (x_k, y_k) of $x^2 - 3y^2 = -2$ with $y_k \leq 31$ are $(x_k, y_k) = (1, 1), (5, 3), (19, 11)$ (which indeed have relatively prime coordinates). These correspond to the values of n: $n = 1, 9, 121$.

- Suppose $g = 2$. Then $n = 2y^2$ and $3n - 2 = 2x^2$ for some relatively prime positive integers x and y. This leads us to the generalized Pell equation $x^2 - 3y^2 = -1$. Again, we set $u = 2 + \sqrt{3}$. We now look for the positive integer solutions (x, y) of $x^2 - 3y^2 = -1$ with $x \leq (\sqrt{u}+1)/2 \approx 1.47$ and $y \leq (\sqrt{u}+1)/(2\sqrt{3}) \approx 0.85$. It turns out that there are no such solutions on these bounds, so by the above lemma, we conclude that $x^2 - 3y^2 = -1$ has no solutions in positive integers.

Hence, the only positive integers $n < 1000$ for which a_n is a rational number are $n = 1, 9, 121$ and the sum is $1 + 9 + 121 = \boxed{131}$.

[1]For proof, see Theorem 3.3 from `https://kconrad.math.uconn.edu/blurbs/ugradnumthy/pelleqn2.pdf`.