PHILIPPINE MATHEMATICAL OLYMPIAD

MCMLXXXIV

PMO

23

QUALIFYING STAGE
FEBRUARY 20 2021

NATIONAL STAGE
MARCH 19 2021 AND MARCH 20 2021

AWARDING CEREMONY
MARCH 21 2021

ABOUT THE PMO

First held in **1984**, the Philippine Mathematical Olympiad (PMO) was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are:

(1) to stimulate the improvement of mathematics education in the country by awakening greater interest in and appreciation of mathematics among students and teachers, and gaining insights into the levels of mathematical learning;
(2) to identify and motivate the mathematically gifted;
(3) to identify potential participants to the International Mathematical Olympiad;
(4) to provide a vehicle for the professional growth of teachers; and
(5) to encourage the involvement of both public and private sectors in the concerted promotion and development of mathematics education.

The PMO is only the first part of the selection program implemented by the Mathematical Society of the Philippines towards the country's participation in the **International Mathematical Olympiad (IMO).** The thirty national finalists of the PMO will be invited to the **Mathematical Olympiad Summer Camp (MOSC),** a training program where participants will experience problem solving at a level that will help them grow in mathematical maturity, in preparation for the IMO. The selection tests and quizzes will then determine the six contestants who will form the country's National Team in Mathematics - the Philippine Team to the International Mathematical Olympiad.

The Philippine Mathematical Olympiad, the Mathematical Olympiad Summer Camp, and the country's participation in the International Mathematical Olympiad, are projects of both the **Mathematical Society of the Philippines** and the **Department of Science and Technology - Science Education Institute**.

The PMO this year is the twenty-third since 1984. The COVID-19 pandemic has made it necessary to hold the PMO **online.** This is the first time a major math competition for high school students with a nationwide scope was held in this online mode. **Two thousand three hundred seventy-nine (2,379)** contestants from Grades 7 to 12 participated in the Qualifying Stage last February 20, from which our **thirty national finalists** and **special awardees** were chosen. From among them will be selected the Philippine Team to **the 62nd International Mathematical Olympiad**, slated to be held from **July 14 to 24, 2021**, in **Saint Petersburg, Russia**.

MESSAGE

My warmest greetings to the participants of the country's most prestigious mathematical competition!

Much has changed in the world in a span of a year due to the COVID-19 pandemic. While our strategies may have been altered, our macro and micro goals remain, and the same is true with our goal of ascending in the global mathematics stage. Momentous as the past decade may be for the Philippines in building credentials in the world stage, there are rooms to grow and new horizons to reach. Hence, it always brings us energy to begin each year with the Philippine Mathematical Olympiad (PMO).

In its rich 23 years, the PMO served as home to the most outstanding mathematics talents in the country and the world. Its goal of awakening greater interest and appreciation in the field of mathematics goes beyond gifted students and extends to their support group and well over to general population. The science department welcomes this as it reinforces our cause of building a culture of science across the nation. Without a doubt, the PMO pushes not just the participants and their teams but also the government and private sectors.

This year's national stage is sure to be more challenging as we all live in a tough time in history. But as history has proven repeatedly, we as Filipinos will soldier on and continue to find ways to excel. Truly, we can sustain our momentum not just in preparing and winning in the International Mathematical Olympiad (IMO), but more so in building the said culture of science for the national good.

The whole Department of Science and Technology – Science Education Institute (DOST-SEI) congratulates the Mathematical Society of the Philippines, our finalists, and all students and parents who made the 23rd PMO a success.

We wish our participants all the best.

JOSETTE T. BIYO
Director, Science Education Institute
Department of Science and Technology

MESSAGE

It has been a year since the NCR and other key areas in the Philippines have been put in community quarantine. About a year ago, our teams of organizers for the Mathematical Olympiad Summer Camp (MOSC) and the Philippine Mathematical Olympiad (PMO) had to carefully think and plan what could be done to continue with our projects during this pandemic. The MOSC, under the leadership of Dr. Christian Chan Shio, shifted to the virtual mode of training our students to prepare for the 61st International Mathematical Olympiad (IMO) scheduled in July 2020. The 23rd PMO, with Director Dr. Richard Eden, was also planned to be held in an online or virtual format.

In the 61st IMO, the Philippine Team got a gold medal (the Philippines' 4th since 2016), two bronze medals and three honorable mentions. An achievement that that we could all be truly proud of even in this time of the pandemic. The 62nd IMO will be held either following the 61 IMO's virtual format or the usual mode where the participants fly out to the host country (Russia). The 23rd PMO is a crucial stage in the selection of our team for the 62nd IMO that will happen in July 14 – 24, 2021. I wish all our national finalists the best as they participate in this important project of the Mathematical Society of the Philippines (MSP) and the Department of Science and Technology – Science Education Institute (DOST – SEI).

The support of the DOST–SEI for the MSP to conduct the PMO, the MOSC and the country's participation in the IMO has surely taken us a long way in the promotion of mathematics and in developing young mathematical talents. Our heartfelt gratitude goes to our friends from DOST – SEI especially to Dr. Ruby Roan Cristobal and Dr. Josette Biyo, who recently celebrated her birthday. Happy Birthday to Dr. Biyo!

Many thanks as well to SHARP Calculators. Thanks to Itemhound for the smooth online conduct of this event. And thanks, as well to Mathizen.com for hosting the PMO webpage. You have all helped to make the successful conduct of this event possible.

I would also like to acknowledge all the hard work put in by everyone working with the PMO organizing team. This is on top of the already demanding work that they do for their respective home institutions in this particularly challenging time of the pandemic.

Congratulations to all the students who made it to the national finals and to the top scorers in their respective areas or regions. This year, we have 2379 who joined in the Qualifying Stage and 30 made it to the National Stage. Thanks to all participants, parents, coaches, teachers, and school administrators who made this event a momentous success.

I wish that everyone will continue to be safe and healthy.
All the best!

EMMANUEL A. CABRAL
President
Mathematical Society of the Philippines

THE PMO TEAM

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Region 11
Joseph Belida
Region 13
Miraluna Herrera
NCR
Kristine Joy Carpio
Raymond Joseph Fadri

SPECIAL AWARDS

Top Contestant Per Region from the Qualifying Stage

NCR
Bryce Ainsley Sanchez
Grace Christian College

CAR
Juan Miguel Sebastian Orille
Philippine Science High School
– Cordillera Administrative Region Campus

REGION 1
Phylline Cristel Calubayan
BHC Educational Institution, Inc.

REGION 2
Jesse Achiles Sanguir
Pattao National High School
Jay Lord Ugaddan
Regional Science High School for Region 2

REGION 3
Enzo Rafael Chan
Bayanihan Institute

REGION 4A
Charles Dwight Pelaez
Quezon Science High School

REGION 4B
Keane Mikah Guinto
Palawan State University

REGION 5
Adrian Richard Salazar
Philippine Science High School
– Bicol Region Campus

REGION 6
Jonathan Anacan
Philippine Science High School
– Western Visayas Campus

REGION 7
Justin Nathaniel Lim
University of San Carlos

REGION 8
Raven Mark Blanca
Philippine Science High School
– Eastern Visayas Campus

REGION 9
Stephen James Ty
Zamboanga Chong Hua High School

REGION 10
Mohammad Nur Casib
Philippine Science High School
– Central Mindanao Campus

REGION 11
Cassidy Kyler Tan
Davao Christian High School

REGION 12
Aniela Roselle Alviola
Dole Philippines School

REGION 13
Joseph Banaybanay
Bayugan National Comprehensive
High School

BARMM
Mohammad Miadh Angkad
Albert Einstein School, Inc.

SPECIAL AWARDS

Top Female Contestant from the Qualifying Stage

Deanne Gabrielle Algenio
Ateneo de Manila Senior High School

Top Junior Contestant from the Qualifying Stage

Jerome Austin Te
Jubilee Christian Academy
Grade 7

Top Contestants Per Area from the Qualifying Stage

LUZON

1st Charles Dwight Pelaez
Quezon Science High School

2nd Adrian Richard Salazar
Philippine Science High School – Bicol Region Campus

3rd Enzo Rafael Chan
Bayanihan Institute

VISAYAS

1st Jonathan Anacan
Philippine Science High School – Western Visayas Campus

2nd Christopher James Yap
St. John's Institute

3rd Justin Nathaniel Lim
University of San Carlos

MINDANAO

1st Stephen James Ty
Zamboanga Chong Hua High School

2nd Mohammad Nur Casib
Philippine Science High School – Central Mindanao Campus

3rd Mohammad Miadh Angkad
Albert Einstein School, Inc.

NCR

1st Bryce Ainsley Sanchez
Grace Christian College

2nd Steven Reyes
Saint Jude Catholic School

3rd Vincent Dela Cruz
Valenzuela City School of Mathematics and Science

3rd Daryll Carlsten Ko
St. Stephen's High School

NATIONAL FINALISTS

Jose Lorenzo Abad

Philippine Science High School – Main Campus

Deanne Gabrielle Algenio

Ateneo de Manila Senior High School

Immanuel Josiah Balete

St. Stephen's High School

Dominic Lawrence Bermudez

Philippine Science High School – Main Campus

Jose Maria Bernardo II

Ateneo de Manila Senior High School

Sarji Elijah Bona

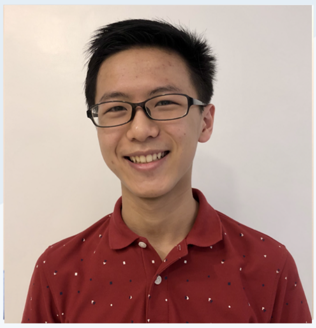
De La Salle University Integrated School Manila

Mohammad Nur Casib

Philippine Science High School – Central Mindanao Campus

Al Patrick Castro

De La Salle University Integrated School Manila



Shawn Darren Chua

MGC New Life Christian Academy

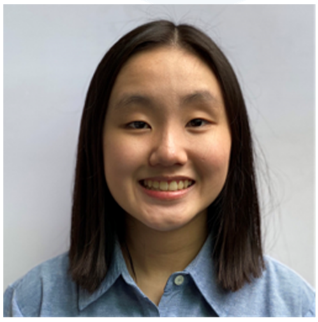
Raphael Dylan Dalida

Philippine Science High School – Main Campus

NATIONAL FINALISTS


Vincent Dela Cruz
Valenzuela City School of Mathematics and Science


Alvann Walter Paredes Dy
Saint Jude Catholic School


Vanessa Ryanne Julio
Ateneo de Manila Senior High School


Daryll Carlsten Ko
St. Stephen's High School


Enrico Rolando Martinez
Philippine Science High School – Main Campus


Citrei Kim Padayao
Quezon City Science High School


Wesley Gavin Palomar
Philippine Science High School – Main Campus


Charles Dwight Pelaez
Quezon Science High School


Steven Reyes
Saint Jude Catholic School


Adrian Richard Salazar
Philippine Science High School – Bicol Region Campus

NATIONAL FINALISTS

Bryce Ainsley Sanchez
Grace Christian College

Oniluv Troy Tabujara
Ateneo de Manila Senior High School

Kiel Sam Talosig
Makati Science High School

Sean Matthew Tan
Jubilee Christian Academy

Keean Nathaniel Tang
Ateneo de Manila Senior High School

Jerome Austin Te
Jubilee Christian Academy

Kristen Steffi Teh
Grace Christian College

Stephen James Ty
Zamboanga Chong Hua High School

Issam Wang
Manila Science High School

Filbert Ephraim Wu
Victory Christian International School

23rd Philippine Mathematical Olympiad

Qualifying Stage, 20 February 2021

PART I. Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

1. In a convex polygon, the number of diagonals is 23 times the number of its sides. How many sides does it have?

(a) 46 (b) 49 (c) 66 (d) 69

2. What is the smallest real number a for which the function $f(x) = 4x^2 - 12x - 5 + 2a$ will always be nonnegative for all real numbers x?

(a) 0 (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 7

3. In how many ways can the letters of the word *PANACEA* be arranged so that the three *A*s are not all together?

(a) 540 (b) 576 (c) 600 (d) 720

4. How many ordered pairs of positive integers (x, y) satisfy $20x + 21y = 2021$?

(a) 4 (b) 5 (c) 6 (d) infinitely many

5. Find the sum of all k for which $x^5 + kx^4 - 6x^3 - 15x^2 - 8k^3x - 12k + 21$ leaves a remainder of 23 when divided by $x + k$.

(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{5}{8}$ (d) $\frac{3}{4}$

6. In rolling three fair twelve-sided dice simultaneously, what is the probability that the resulting numbers can be arranged to form a geometric sequence?

(a) $\frac{1}{72}$ (b) $\frac{5}{288}$ (c) $\frac{1}{48}$ (d) $\frac{7}{288}$

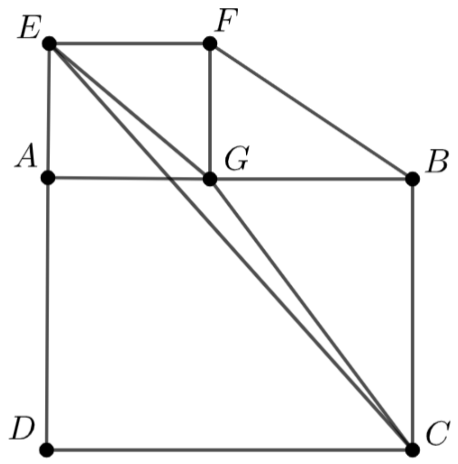
7. How many positive integers n are there such that $\frac{n}{120 - 2n}$ is a positive integer?

(a) 2 (b) 3 (c) 4 (d) 5

8. Three real numbers a_1, a_2, a_3 form an arithmetic sequence. After a_1 is increased by 1, the three numbers now form a geometric sequence. If a_1 is a positive integer, what is the smallest positive value of the common difference?

(a) 1 (b) $\sqrt{2} + 1$ (c) 3 (d) $\sqrt{5} + 2$

9. Point G lies on side AB of square $ABCD$ and square $AEFG$ is drawn outwards $ABCD$, as shown in the figure below. Suppose that the area of triangle EGC is $1/16$ of the area of pentagon $DEFBC$. What is the ratio of the areas of $AEFG$ and $ABCD$?



(a) $4 : 25$ (b) $9 : 49$ (c) $16 : 81$ (d) $25 : 121$

10. In how many ways can 2021 be written as a sum of two or more consecutive integers?

(a) 3 (b) 5 (c) 7 (d) 9

11. In quadrilateral $ABCD$, $\angle CBA = 90°$, $\angle BAD = 45°$, and $\angle ADC = 105°$. Suppose that $BC = 1 + \sqrt{2}$ and $AD = 2 + \sqrt{6}$. What is the length of AB?

(a) $2\sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $3 + \sqrt{2}$ (d) $3 + \sqrt{3}$

12. Alice tosses two biased coins, each of which has a probability p of obtaining a head, simultaneously and repeatedly until she gets two heads. Suppose that this happens on the rth toss for some integer $r \geq 1$. Given that there is 36% chance that r is even, what is the value of p?

(a) $\dfrac{\sqrt{7}}{4}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{\sqrt{2}}{2}$ (d) $\dfrac{3}{4}$

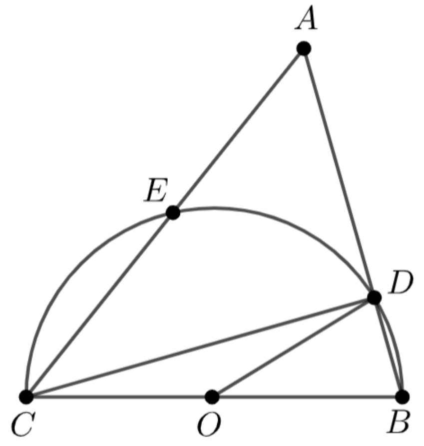
13. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t and $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t. How many real numbers x between 1 and 23 satisfy $\lfloor x \rfloor \{x\} = 2\sqrt{x}$?

(a) 18 (b) 19 (c) 20 (d) 21

14. Find the remainder when $\displaystyle\sum_{n=2}^{2021} n^n$ is divided by 5.

(a) 1 (b) 2 (c) 3 (d) 4

15. In the figure below, BC is the diameter of a semicircle centered at O, which intersects AB and AC at D and E respectively. Suppose that $AD = 9$, $DB = 4$, and $\angle ACD = \angle DOB$. Find the length of AE.



(a) $\frac{117}{16}$ (b) $\frac{39}{5}$ (c) $2\sqrt{13}$ (d) $3\sqrt{13}$

PART II. All answers are positive integers. Do not use commas if there are more than 3 digits, e.g., type 1234 instead of 1,234. A positive fraction a/b is in lowest terms if a and b are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. Consider all real numbers c such that $|x - 8| + |4 - x^2| = c$ has exactly three real solutions. The sum of all such c can be expressed as a fraction a/b in lowest terms. What is $a + b$?

17. Find the smallest positive integer n for which there are exactly 2323 positive integers less than or equal to n that are divisible by 2 or 23, but not both.

18. Let $P(x)$ be a polynomial with integer coefficients such that $P(-4) = 5$ and $P(5) = -4$. What is the maximum possible remainder when $P(0)$ is divided by 60?

19. Let $\triangle ABC$ be an equilateral triangle with side length 16. Points D, E, F are on CA, AB, and BC, respectively, such that $DE \perp AE$, $DF \perp CF$, and $BD = 14$. The perimeter of $\triangle BEF$ can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

20. How many subsets of the set $\{1, 2, 3, \ldots, 9\}$ do not contain consecutive odd integers?

21. For a positive integer n, define $s(n)$ as the smallest positive integer t such that n is a factor of $t!$. Compute the number of positive integers n for which $s(n) = 13$.

22. Alice and Bob are playing a game with dice. They each roll a die six times, and take the sums of the outcomes of their own rolls. The player with the higher sum wins. If both players have the same sum, then nobody wins. Alice's first three rolls are 6, 5, and 6, while Bob's first three rolls are 2, 1, and 3. The probability that Bob wins can be written as a fraction a/b in lowest terms. What is $a + b$?

23. Let $\triangle ABC$ be an isosceles triangle with a right angle at A, and suppose that the diameter of its circumcircle Ω is 40. Let D and E be points on the arc BC not containing A such that D lies between B and E, and AD and AE trisect $\angle BAC$. Let I_1 and I_2 be the incenters of $\triangle ABE$ and $\triangle ACD$ respectively. The length of I_1I_2 can be expressed in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

24. Find the number of functions f from the set $S = \{0, 1, 2, \ldots, 2020\}$ to itself such that, for all $a, b, c \in S$, all three of the following conditions are satisfied:

(i) If $f(a) = a$, then $a = 0$;

(ii) If $f(a) = f(b)$, then $a = b$; and

(iii) If $c \equiv a + b \pmod{2021}$, then $f(c) \equiv f(a) + f(b) \pmod{2021}$.

25. A sequence $\{a_n\}$ of real numbers is defined by $a_1 = 1$ and for all integers $n \geq 1$,

$$a_{n+1} = \frac{a_n\sqrt{n^2+n}}{\sqrt{n^2+n+2a_n^2}}.$$

Compute the sum of all positive integers $n < 1000$ for which a_n is a rational number.

Answers

Part I. (2 points each)

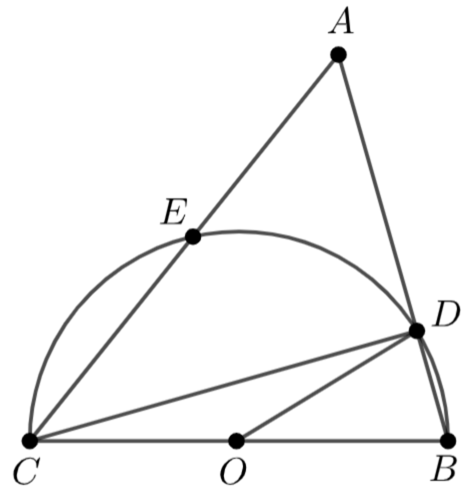
1. B
2. D
3. D
4. B
5. B
6. D
7. B
8. B
9. A
10. C
11. C
12. A
13. A
14. D
15. B

Part II. (5 points each)

16. 93
17. 4644
18. 41
19. 31
20. 208
21. 792
22. 3895
23. 20
24. 1845
25. 131

Solutions to selected problems:

15. In the figure below, BC is the diameter of a semicircle centered at O, which intersects AB and AC at D and E respectively. Suppose that $AD = 9$, $DB = 4$, and $\angle ACD = \angle DOB$. Find the length of AE.



Solution. Let $\angle DOB = \angle EOC = \alpha$. Note that $\angle DCB = \frac{\alpha}{2}$. Also, note that $\tan\frac{\alpha}{2} = \frac{4}{DC}$ and $\tan\alpha = \frac{9}{DC} = \frac{9}{4}\tan\frac{\alpha}{2}$. Let $x = \tan\frac{\alpha}{2}$. By the double-angle formula,

$$\frac{9}{4}x = \frac{2x}{1-x^2}$$
$$\frac{9}{4}x - \frac{9}{4}x^3 = 2x$$
$$\frac{1}{4}x\left(1-9x^2\right) = 0$$

and thus $x = 0$ or $x = \pm\frac{1}{3}$. Clearly, only $x = \frac{1}{3}$ is possible here. Thus, $CD = 12$. Note also that $\angle CDB = \angle ADC = 90°$, and so by the Pythagorean theorem, $AC = \sqrt{9^2 + 12^2} = 15$.

Finally, by the power of a point theorem, we have $AE \cdot AC = AD \cdot AB$ and so $AE \cdot 15 = 9(9+4)$, which gives us $AE = \frac{117}{15} = \boxed{\frac{39}{5}}$.

17. Find the smallest positive integer n for which there are exactly 2323 positive integers less than or equal to n that are divisible by 2 or 23, but not both.

Solution. The number of positive integers from 1 to n that are divisible by 2 or 23, but not both, is

$$f(n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{23} \right\rfloor - 2\left\lfloor \frac{n}{46} \right\rfloor.$$

We need to find $f(n) = 2323$, which can be done by some trial and error.

Note that if n is a multiple of 2 and 23, then the floor divisions are non-rounded exact divisions, in which case,

$$\frac{n}{2} + \frac{n}{23} - 2\,\frac{n}{46} = \frac{n}{2}.$$

So, for example, $f(4646) = 2323$. We can then just tick downwards.

Since 4646 and 4645 do not satisfy our criteria, we know that $f(4646) = f(4645) = f(4644)$, i.e. we can remove 4646 and 4645 and the count does not go down (we weren't counting them). However, 4644 *does* satisfy our criteria, so $f(4643) = 2322$.

Note that by definition, this function is non-decreasing. Thus, we conclude that $n = \boxed{4644}$ is the first n that satisfies the given conditions.

18. Let $P(x)$ be a polynomial with integral coefficients such that $P(-4) = 5$ and $P(5) = -4$. What is the maximum possible remainder when $P(0)$ is divided by 60?

Solution.

$$0 - (-4)|P(0) - P(-4) \text{ or } 4|P(0) - 5, \text{ so } P(0) \equiv 5 (\text{mod } 4)$$

$$5 - 0|P(5) - P(0) \text{ or } 5| - 4 - P(0), \text{ so } P(0) \equiv -4 (\text{mod } 5)$$

By the Chinese Remainder Theorem, there is a solution r that satisfies both of the previous equations, and this solution is unique modulo $4 \cdot 5 = 20$. It is easy to verify that this solution is 1. Thus, $P(0) \equiv 1 (\text{mod } 20)$. This implies that $P(0)$ can be $1, 21$, or $41 (\text{mod } 60)$. The largest remainder $\boxed{41}$ is indeed achievable, for example by the polynomial $-2(x+4)(x-5)+1-x$.

19. Let $\triangle ABC$ be an equilateral triangle with side length 16. Points D, E, F are on CA, AB, and BC, respectively, such that $DE \perp AE$, $DF \perp CF$, and $BD = 14$. The perimeter of $\triangle BEF$ can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

Solution. Let $\overline{AD} = 2x$, then $\overline{DC} = 16 - 2x$. Since $\triangle DAE$ and $\triangle DCF$ are both 30-60-90 triangles, then $\overline{AE} = \overline{AD}/2 = x$, $\overline{ED} = x\sqrt{3}$ and $\overline{CF} = \overline{DC}/2 = 8 - x$, $\overline{FD} = (8 - x)\sqrt{3}$. Since $\overline{AB} = \overline{BC} = 16$, then $\overline{EB} = 16 - x$ and $\overline{FB} = 8 + x$. Using Pythagorean Theorem on $\triangle DEB$, we have

$$\begin{aligned}
\overline{ED}^2 + \overline{EB}^2 &= \overline{BD}^2 \\
(x\sqrt{3})^2 + (16 - x)^2 &= 14^2 \\
3x^2 + 256 - 32x + x^2 &= 196 \\
x^2 - 8x + 15 &= 0 \\
(x - 5)(x - 3) &= 0 \\
x &= 5, 3
\end{aligned}$$

Choosing either of the two values of x will give the same result for the perimeter of $\triangle BEF$. Suppose we choose $x = 5$, then $\overline{EB} = 11$ and $\overline{FB} = 13$. By Cosine Law on $\triangle BEF$, we have

$$\begin{aligned}
\overline{EF}^2 &= \overline{EB}^2 + \overline{FB}^2 - 2 \cdot \overline{EB} \cdot \overline{FB} \cos 60^\circ \\
\overline{EF}^2 &= 11^2 + 13^2 - 2(11)(13)(1/2) \\
\overline{EF} &= \sqrt{121 + 169 - 143} = 7\sqrt{3}
\end{aligned}$$

Thus, the perimeter of $\triangle BEF = 24 + 7\sqrt{3}$, which means that $a + b + c + d = 24 + 0 + 7 + 0 = \boxed{31}$.

20. How many subsets of the set $\{1, 2, 3, \ldots, 9\}$ do not contain consecutive odd integers?

Solution. Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain consecutive odd integers. We work on the following cases depending on whether such a subset contains 1 or not:

- If such a subset does not contain 1, then its elements must consist of elements from $\{2, 3, \ldots, n\}$. The number of subsets with no element smaller than 3 is a_{n-2}; for each of these subsets, we include 2 as well, giving another a_{n-2} subsets. Thus, there are $2a_{n-2}$ subsets in this case.
- If such a subset contains 1, then it must not contain 3 and its elements must consist of elements from $\{1, 2, 4, 5, \ldots, n\}$. The number of subsets with no element smaller than 5 is a_{n-4}; for each of these subsets, we take the union with a non-empty subset of $\{2, 4\}$ as well, giving another $3a_{n-4}$ subsets. Thus, there are $4a_{n-4}$ subsets in this case.

We now obtain the recurrence formula $a_n = 2a_{n-2} + 4a_{n-4}$. With $a_1 = 2, a_2 = 4, a_3 = 6$ and $a_4 = 12$, routine computation gives $a_9 = \boxed{208}$.

21. For a positive integer n, define $s(n)$ as the smallest positive integer t such that n divides $t!$. Compute the number of positive integers n for which $s(n) = 13$.

Solution. For a positive integer k, consider the set $A(k) = \{n \in \mathbb{N} : s(n) = k\}$ and we wish to find $\#A(13)$. From the definition of $s(n)$, any element of $A(k)$ must divide $k!$ but not $(k-1)!$. Thus, any element of $A(k)$ must be a divisor of $k!$ that is not a divisor of $(k-1)!$. We see that $\#A(k) = \sigma_0(k!) - \sigma_0((k-1)!)$, where $\sigma_0(n)$ is the number of divisors of n, so that $\#A(13) = \sigma_0(13!) - \sigma_0(12!)$.

Note that for any prime p, the highest power of p that divides $k!$ is p^e, where $e = \sum_{j=1}^{\infty} \lfloor n/p^j \rfloor$. Using this formula, we determine the prime factorization of 13!: $13! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1$. As $13! = 13 \cdot 12!$, we get $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$. Hence, we obtain

$$\#A(13) = 11 \cdot 6 \cdot 3 \cdot 2^3 - 11 \cdot 6 \cdot 3 \cdot 2^2 = 198 \cdot 4 = \boxed{792}.$$

22. Alice and Bob are playing a game with dice. They each roll a die six times, and take the sums of the values of their own rolls. The player with the higher sum wins. If both players have the same sum, then nobody wins. Alice's first three rolls are 6, 5, and 6, while Bob's first three rolls are 2, 1, and 3. The probability that Bob wins can be written as a fraction a/b in lowest terms. What is $a + b$?

Solution. Let a_i denote the value of Alice's ith roll, and b_i denote the value of Bob's ith roll. For Bob to win, the following inequality must hold:

$$6 + 5 + 6 + a_4 + a_5 + a_6 < 2 + 1 + 3 + b_4 + b_5 + b_6$$

Rearranging yields

$$(a_4 - 1) + (a_5 - 1) + (a_6 - 1) + (6 - b_4) + (6 - b_5) + (6 - b_6) < 4.$$

Let $x_i = a_{i+3} - 1$ and $x_{i+3} = 6 - b_{i=3}$ for $i = 1, 2, 3$. The inequality then simplifies to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 4.$$

We claim that each nonnegative integer solutions to this inequality correspond to a valid situation of dice rolls. Note that the previous inequality implies that $0 \le x_i < 4$, so $1 \le x_i + 1 < 5$ and $2 < 6 - x_i \le 6$. Thus, the corresponding dice rolls for both Alice and Bob are within the bounds.

The expression on the left can have a value of either $0, 1, 2,$ or 3. Thus, the number of nonnegative integer solutions of the aforementioned equation is $\binom{5}{5} + \binom{6}{5} + \binom{7}{5} + \binom{8}{5} = 1 + 6 + 21 + 56 = 84$. Thus, the probability that Bob wins is $\dfrac{84}{6^6} = \dfrac{7}{3888}$, and so $a + b = 7 + 3888 = \boxed{3895}$.

23. Let $\triangle ABC$ be an isosceles triangle with a right angle at A, and suppose that the diameter of its circumcircle Ω is 40. Let D and E be points on the arc BC not containing A such that D lies between B and E, and AD and AE trisect $\angle BAC$. Let I_1 and I_2 be the incenters of $\triangle ABE$ and $\triangle ACD$ respectively. The length of I_1I_2 can be expressed in the form $a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}$, where a, b, c, and d are integers. Find $a+b+c+d$.

Solution. Let O be the center of Ω. Note that $\angle OBD = \angle CBD = \frac{\pi}{3}$, so $\triangle OBD$ is an equilateral triangle. Thus, $BD = BO = \frac{4}{2} = 2$. This implies that $EC = DE = BD = 2$. Clearly, I_1 and I_2 lie on the segments AD and AE respectively. It is well-known that $DI_1 = DB$, so $DI_1 = 20$. Similarly, $EI_2 = 20$. Now, note that $\angle EDI_1 = \angle EDC + \angle CDA = 30° + 45° = 75°$. Similarly, $\angle DEI_2 = 75°$. Looking at the isosceles trapezoid I_1I_2ED, the length of I_1I_2 must then be $DE - DI_1\cos 75° - EI_2 \cos 75° = 2 - 40\cos 75° = 20 - 40\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = 20 - \sqrt{6} + \sqrt{2}$, and so $a+b+c+d = 20+10+0-10 = \boxed{20}$.

24. How many functions f are there from the set $S = \{0, 1, 2, \ldots, 2020\}$ to itself such that, for all $a, b, c \in S$, all three of the following conditions are satisfied:

(i) If $f(a) = a$, then $a = 0$;

(ii) If $f(a) = f(b)$, then $a = b$; and

(iii) If $c \equiv a + b \pmod{2021}$, then $f(c) \equiv f(a) + f(b) \pmod{2021}$.

Solution. Note that, from (i), our function is completely determined by $f(1)$; i.e., $f(a) \equiv af(1) \pmod{2021}$. Then, from (i) and (ii), we need that $f(a) \neq 0$ if $a \neq 0$; otherwise, if $a \neq 0$ but $f(a) = 0$, $f(b) = f(a+b)$ for any b. Thus, if $a \neq 0$, we need that $af(1) \neq 0 \pmod{2021}$ for any $a \in S \setminus \{0\}$. If $\gcd(f(1), 2021) = d > 1$, then note that $a = \frac{2021}{d} \in S \setminus \{0\}$, and from (i), $f(a) \equiv af(1) \equiv 2021 \equiv 0 \pmod{2021}$. As we just established, this is not allowed. Hence, $\gcd(f(1), 2021) = 1$.

Moreover, from (iii), we need that $f(a) = af(1) \not\equiv a \pmod{2021}$ if $a \neq 0$; in other words, $2021 \nmid a(f(1)-1)$ if $a \neq 0$. By similar reasoning to earlier, suppose that $\gcd(f(1)-1, 2021) = d > 1$. Then $a = \frac{2021}{d} \in S \setminus \{0\}$, and $2021 \mid a(f(1)-1)$; thus, $f(a) = a$. We thus need that $\gcd(f(1)-1, 2021) = 1$ as well.

We count the number of integers that satisfy both these conditions. We use complementary counting here; thus, we start by counting those that fail to satisfy at least one condition. Indeed, we count the number of values of $f(1)$ that are *not* coprime to 2021. Either they are divisible by 43, or 47, or both. Since only 0 is divisible by both, by the principle of inclusion and exclusion, there are $47 + 43 - 1 = 89$ possible values of $f(1)$ so that $f(1)$ is not coprime to 2021. By the same reasoning, there are also 89 possible values of $f(1)$ such that $f(1) - 1$ is not coprime to 2021. Finally, we count the values of $f(1)$ for which neither $f(1)$ nor $f(1) - 1$ is coprime to 2021. This happens precisely when $43 \mid f(1)$ and $47 \mid f(1) - 1$, or $47 \mid f(1)$ and $43 \mid f(1) - 1$. By the Chinese remainder theorem, each of these possibilities gives one value of $f(1)$. Thus, by the principle of inclusion and exclusion, there are $2 \cdot 89 - 2 = 176$ such values of $f(1)$ that fail to satisfy at least one condition. This gives us $2021 - 176 = \boxed{1845}$ possible values of $f(1)$, and thus possible functions f.

25. A sequence $\{a_n\}$ of real numbers is defined by $a_1 = 1$ and $a_{n+1} = \dfrac{a_n\sqrt{n^2+n}}{\sqrt{n^2+n+2a_n^2}}$ for all integers $n \geq 1$. Compute the sum of all positive integers $n < 1000$ for which a_n is a rational number.

Solution. First, note that for $k \geq 1$,

$$a_{k+1}^2 = \frac{k(k+1)a_k^2}{k(k+1)+2a_k^2} \iff \frac{1}{a_{k+1}^2} - \frac{1}{a_k^2} = \frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1}$$

and summing the second equation from $k=1$ to $k=n-1$ with $n \geq 2$, we get

$$\frac{1}{a_n^2} - \frac{1}{a_1^2} = \sum_{k=1}^{n-1} \left(\frac{1}{a_{k+1}^2} - \frac{1}{a_k^2} \right) = \sum_{k=1}^{n-1} \left(\frac{2}{k} - \frac{2}{k+1} \right) = 2 - \frac{2}{n}.$$

Since $a_1 = 1$, we see that

$$\frac{1}{a_n^2} = 3 - \frac{2}{n} \iff a_n = \sqrt{\frac{n}{3n-2}}$$

for all integers $n \geq 1$ and we wish to find the sum of all positive integers $n < 1000$ such that $\frac{n}{3n-2}$ is a square of some rational number. To help us look for such integers n, we use the following lemma that provides integer solutions to the generalized Pell equation.

Lemma.[1] *Let d be a squarefree positive integer, and let a and b be positive integers such that $a^2 - db^2 = 1$. Set $u = a + b\sqrt{d}$. Then for each nonzero integer n, every solution of $x^2 - dy^2 = n$ is a power of u times $x + y\sqrt{d}$ where (x, y) is an integer solution of $x^2 - dy^2 = n$ with $|x| \leq \sqrt{|n|}(\sqrt{u}+1)/2$ and $|y| \leq \sqrt{|n|}(\sqrt{u}+1)/(2\sqrt{d})$.*

We now let $g = \gcd(n, 3n-2)$. Then $g \in \{1, 2\}$ since $g \mid 3n - (3n-2) = 2$. We now consider the following cases:

- Suppose $g = 1$. Then $n = y^2$ and $3n - 2 = x^2$ for some relatively prime positive integers x and y. This leads us to the generalized Pell equation $x^2 - 3y^2 = -2$. Set $u = 2 + \sqrt{3}$, with $(2, 1)$ being a solution of $x^2 - 3y^2 = 1$ in positive integers. We now look for the positive integer solutions (x, y) of $x^2 - 3y^2 = -2$ with $x \leq \sqrt{2}(\sqrt{u}+1)/2 \approx 2.07$ and $y \leq \sqrt{2}(\sqrt{u}+1)/(2\sqrt{3}) \approx 1.2$. We obtain $(x, y) = (1, 1)$ as the only such integer solution, so by the above lemma, we see that all positive integer solutions (x_k, y_k) of $x^2 - 3y^2 = -2$ are given by $x_k + y_k\sqrt{3} = (1+\sqrt{3})(2+\sqrt{3})^k$ for all integers $k \geq 0$. We now compute this product for small values of k:

k	0	1	2	3
$x_k + y_k\sqrt{3}$	$1+\sqrt{3}$	$5+3\sqrt{3}$	$19+11\sqrt{3}$	$71+41\sqrt{3}$

  As $n < 1000$, we require that $y_k \leq 31$, so the positive integer solutions (x_k, y_k) of $x^2 - 3y^2 = -2$ with $y_k \leq 31$ are $(x_k, y_k) = (1, 1), (5, 3), (19, 11)$ (which indeed have relatively prime coordinates). These correspond to the values of n: $n = 1, 9, 121$.

- Suppose $g = 2$. Then $n = 2y^2$ and $3n - 2 = 2x^2$ for some relatively prime positive integers x and y. This leads us to the generalized Pell equation $x^2 - 3y^2 = -1$. Again, we set $u = 2 + \sqrt{3}$. We now look for the positive integer solutions (x, y) of $x^2 - 3y^2 = -1$ with $x \leq (\sqrt{u}+1)/2 \approx 1.47$ and $y \leq (\sqrt{u}+1)/(2\sqrt{3}) \approx 0.85$. It turns out that there are no such solutions on these bounds, so by the above lemma, we conclude that $x^2 - 3y^2 = -1$ has no solutions in positive integers.

Hence, the only positive integers $n < 1000$ for which a_n is a rational number are $n = 1, 9, 121$ and the sum is $1 + 9 + 121 = \boxed{131}$.

[1]For proof, see Theorem 3.3 from `https://kconrad.math.uconn.edu/blurbs/ugradnumthy/pelleqn2.pdf`.

23^{rd} Philippine Mathematical Olympiad

National Stage - Day 1

19 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

1. In convex quadrilateral $ABCD$, $\angle CAB = \angle BCD$. P lies on line BC such that $AP = PC$, Q lies on line AP such that AC and DQ are parallel, R is the point of intersection of lines AB and CD, and S is the point of intersection of lines AC and QR. Line AD meets the circumcircle of AQS again at T. Prove that AB and QT are parallel.

2. Let n be a positive integer. Show that there exists a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\sum_{k=1}^{n} \frac{k}{(k + \sigma(k))^2} < \frac{1}{2}.$$

3. Denote by $\mathbb{Q}^+$ the set of positive rational numbers. A function $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfies

   - $f(p) = 1$ for all primes p, and
   - $f(ab) = af(b) + bf(a)$ for all $a, b \in \mathbb{Q}^+$.

   For which positive integers n does the equation $nf(c) = c$ have at least one solution c in $\mathbb{Q}^+$?

4. Determine the set of all polynomials $P(x)$ with real coefficients such that the set $\{P(n) \mid n \in \mathbb{Z}\}$ contains all integers, except possibly finitely many of them.

23^{rd} Philippine Mathematical Olympiad

National Stage - Day 2

20 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

5. A positive integer is called *lucky* if it is divisible by 7, and the sum of its digits is also divisible by 7. Fix a positive integer n. Show that there exists some lucky integer ℓ such that $|n-\ell| \leq 70$.

6. A certain country wishes to interconnect 2021 cities with flight routes, which are always two-way, in the following manner:

- There is a way to travel between any two cities either via a direct flight or via a sequence of connecting flights.
- For every pair (A, B) of cities that are connected by a direct flight, there is another city C such that (A, C) and (B, C) are connected by direct flights.

Show that at least 3030 flight routes are needed to satisfy the two requirements.

7. Let a, b, c, and d be real numbers such that $a \geq b \geq c \geq d$ and

$$a+b+c+d=13$$
$$a^2+b^2+c^2+d^2=43.$$

Show that $ab \geq 3+cd$.

8. In right triangle ABC, $\angle ACB = 90°$ and $\tan A > \sqrt{2}$. M is the midpoint of AB, P is the foot of the altitude from C, and N is the midpoint of CP. Line AB meets the circumcircle of CNB again at Q. R lies on line BC such that QR and CP are parallel, S lies on ray CA past A such that $BR = RS$, and V lies on segment SP such that $AV = VP$. Line SP meets the circumcircle of CPB again at T. W lies on ray VA past A such that $2AW = ST$, and O is the circumcenter of SPM. Prove that lines OM and BW are perpendicular.

23RD PMO WINNERS

Bryce Ainsley Sanchez
Grace Christian College
CHAMPION

Steven Reyes
Saint Jude Catholic School
FIRST RUNNER UP

Raphael Dylan Dalida
Philippine Science High School – Main Campus
SECOND RUNNER UP

The prizes for the top three contestants in the National Stage are the following:

CHAMPION - P 20,000, medal, trophy, certificate
FIRST RUNNER-UP - P 15,000, medal, trophy, certificate
SECOND RUNNER-UP - P 10,000, medal, trophy, certificate

Their coaches will receive P 5,000, P 3,000, and P 2,000, respectively, and a certificate.

Their schools will receive a trophy.

Each special awardee will receive a medal. Calculators and gifts from Sharp will also be given to selected awardees.

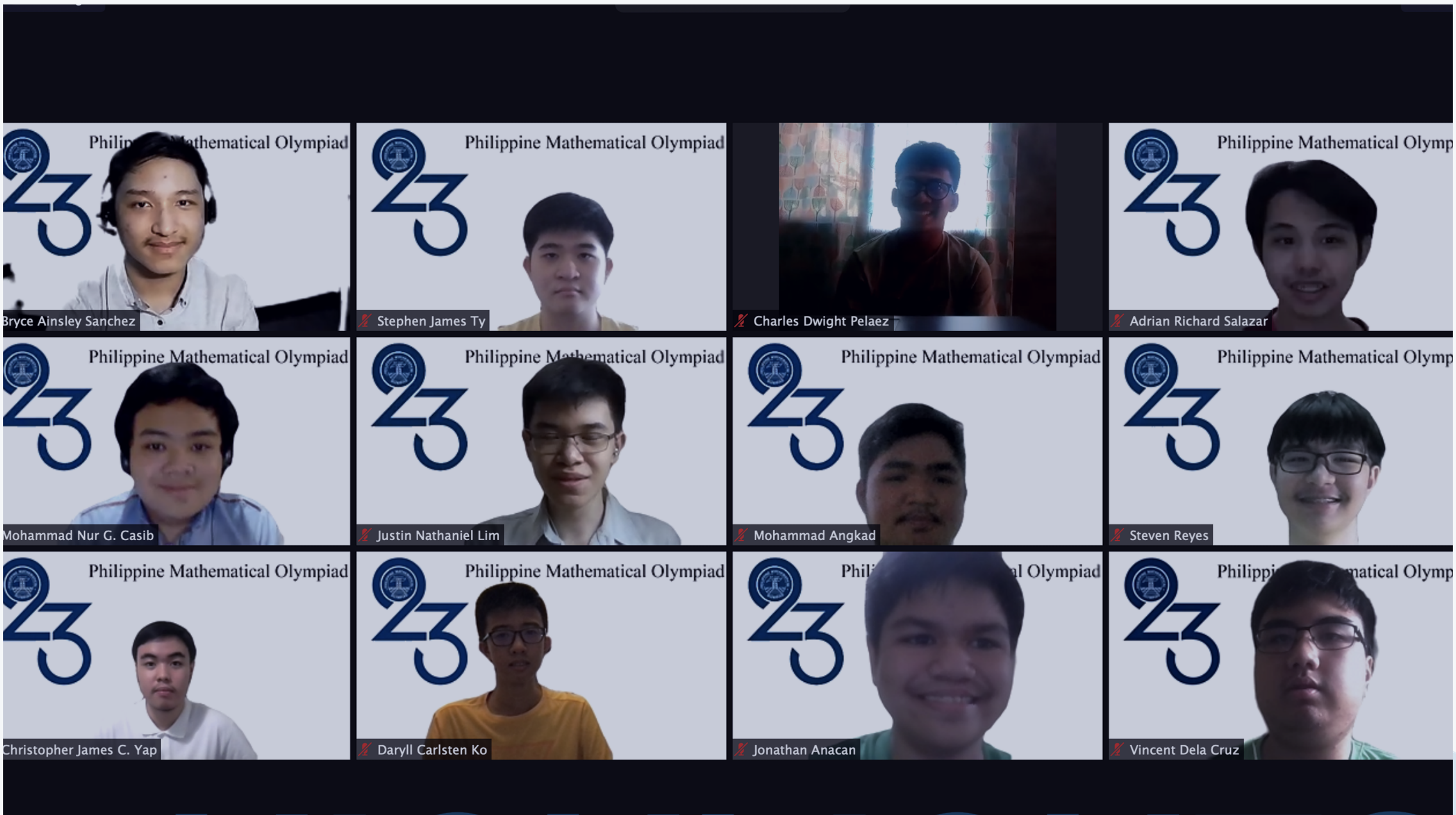
23RD PMO

HIGHLIGHTS

23RD PMO

HIGHLIGHTS

23RD PMO



HIGHLIGHTS

22ND PMO

HIGHLIGHTS

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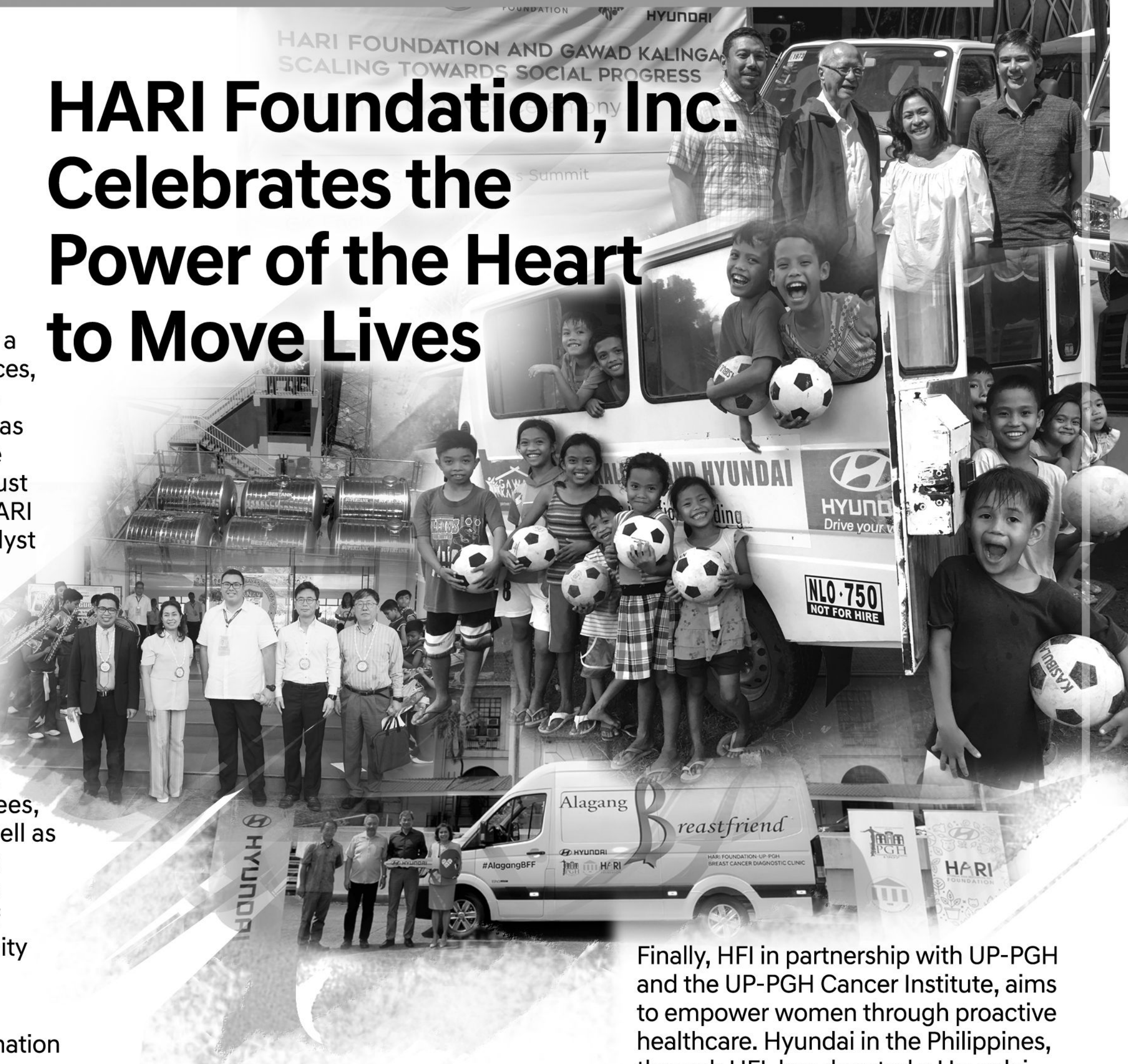
HARI Foundation, Inc. Celebrates the Power of the Heart to Move Lives



From its humble beginnings over a decade ago, Hyundai Asia Resources, Inc. (HARI), the official Philippine distributor of Hyundai vehicles, has earned its place as a leader in the automotive industry. More than just reaping success as a business, HARI has taken upon itself to be a catalyst of change in Philippine society, inspired by the global vision of Hyundai Motor Company (HMC) to build a better world for all.

Through its Corporate Social Responsibility (CSR) arm, HARI Foundation, Inc. (HFI), HARI has engaged employees, dealerships, and customers, as well as partners in the public and private sectors in trail-blazing endeavors across the country in the areas of education, environment, community development, healthcare, and women's and children's rights.

Among HFI's roster of partners in nation building are Gawad Kalinga, St. Scholastica's Priory, the Department of Science and Technology (DOST), National Museum, The Mind Museum, Synergeia Foundation, the ASEAN Center for Biodiversity, and UP Philippine General Hospital.

Fully aware that poverty is more than just the lack of resources, but of options to pursue a better life, HFI has streamlined its efforts to a more targeted, long-term, multi-disciplinary, and multi-stakeholder solution. Education is at the core of HFI's contribution to the Philippine agenda for sustainability.

Says HFI President Ma. Fe Perez-Agudo, "I have always been a believer in empowering people through education. By engaging with like-minded partners from the government and the private sectors, we aim to be the Filipino's lifetime partner in building a sustainable, empowered nation one community at a time."

HFI's over 10 years of working with and learning from its various CSR partners and beneficiaries have led to its developing programs that respond to the needs of our times.

The Hyundai New Thinkers Circuit (HNTC), designed in partnership with the DOST, served as an innovative science literacy program that provokes, fosters, and nurtures leadership and the innovative spirit among outstanding high school students who can take the lead in building a climate change-resilient Philippines. Launched in 2013, HNTC yielded 11 scholars who are pursuing studies in the sciences at the country's top universities.

On March 22, 2017, HFI, Hyundai Motor Company (HMC) Korea, the Institute for Global Education, Exchange and Internship (IGEEI), and the Tanay local government collaborated to build the pilot Rain Water Harvesting System in Rawang Elementary School in Tanay City. This filtration method, an invention of Prof. Han Moo Young of Seoul National University, is capable of producing and storing potable water from rain gathered in roof gutters. Some 200 students of Rawang Elementary School were among its first beneficiaries.This project, which will soon be replicated in other under-served communities in the country, bagged for HFI the 2017 Gold Award from the Society of Philippine Motoring Journalists (SPMJ).

HFI has likewise been an active supporter of Gawad Kalinga (GK) efforts at rehabilitating war-torn Marawi through the donation of vehicles that ply the province as Kusina ng Kalinga (KNK) vans. KNK is GK's platform to address malnutrition among children in public schools, on the streets, and in conflict areas. HFI is taking a step further to help Marawi back to its feet with the rebuilding of public schools in strategic areas of the province.

Finally, HFI in partnership with UP-PGH and the UP-PGH Cancer Institute, aims to empower women through proactive healthcare. Hyundai in the Philippines, through HFI, has donated a Hyundai H350 luxury van customized into a state-of-the-art mobile cancer diagnostic clinic that will transport UP-PGH medical missions to under-served sectors of the country. Dubbed the Alagang Breastfriend project, this comprehensive breast cancer awareness campaign is envisioned to provide women information and access to important resources and technology that would improve their overall well-being, thereby enabling them to lead healthier, more productive lives.

"In a nutshell, the story of HFI in this new century goes by the acronym H.E.A.R.T.—Health, Education, Arts, Rebuilding, and Transformative Leadership. We kick off a new leg in our journey to broaden our reach and design programs that are more meaningful to people, especially at the grassroots", concluded Ms. Agudo.

More than a foundation, HFI represents a movement to build a better world for all, one community at a time, driven by faith in the power of the Filipino heart to innovate, engage, and collaborate to drive shared dreams forward and give rise to generations of leaders capable of advancing sustainability in all important aspects of our lives.

ITEMHOUND

We are a technology-based company who takes pride in its rich background in the execution of **virtual events and web applications**.

We constantly innovate and improve our solutions,
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Recreate the vibe of an on-ground event into virtual showcase. Host webinars, conferences, product launches or trade fairs. Incorporate engaging features and create a fun and memorable experience to your virtual attendees.

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Hold virtual races in the new normal. Promote health and wellness by encouraging people to participate at the comfort of their own place and time. Reward them with freebies delivered directly to their home.

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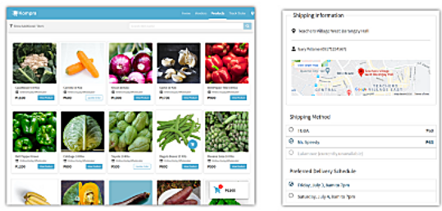
Launch your own online store.
Provide options to your buyers to select their preferred delivery and payment methods. Take an option to have the items fulfilled by our team.



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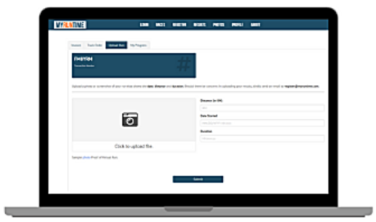
Create an online registration form for your event. Allow interested participants to securely submit their details and payment, with a peace of mind. Get access to see real-time conversions.



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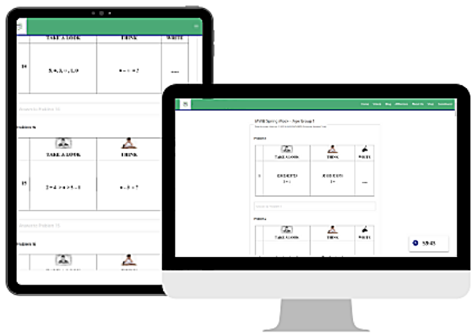

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