23^{rd} Philippine Mathematical Olympiad

National Stage - Day 1

19 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

1. In convex quadrilateral $ABCD$, $\angle CAB = \angle BCD$. P lies on line BC such that $AP = PC$, Q lies on line AP such that AC and DQ are parallel, R is the point of intersection of lines AB and CD, and S is the point of intersection of lines AC and QR. Line AD meets the circumcircle of AQS again at T. Prove that AB and QT are parallel.

2. Let n be a positive integer. Show that there exists a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\sum_{k=1}^{n} \frac{k}{(k + \sigma(k))^2} < \frac{1}{2}.$$

3. Denote by $\mathbb{Q}^+$ the set of positive rational numbers. A function $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfies

   - $f(p) = 1$ for all primes p, and
   - $f(ab) = af(b) + bf(a)$ for all $a, b \in \mathbb{Q}^+$.

   For which positive integers n does the equation $nf(c) = c$ have at least one solution c in $\mathbb{Q}^+$?

4. Determine the set of all polynomials $P(x)$ with real coefficients such that the set $\{P(n) \mid n \in \mathbb{Z}\}$ contains all integers, except possibly finitely many of them.

23^{rd} Philippine Mathematical Olympiad

National Stage - Day 2

20 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

5. A positive integer is called *lucky* if it is divisible by 7, and the sum of its digits is also divisible by 7. Fix a positive integer n. Show that there exists some lucky integer ℓ such that $|n - \ell| \leq 70$.

6. A certain country wishes to interconnect 2021 cities with flight routes, which are always two-way, in the following manner:

- There is a way to travel between any two cities either via a direct flight or via a sequence of connecting flights.
- For every pair (A, B) of cities that are connected by a direct flight, there is another city C such that (A, C) and (B, C) are connected by direct flights.

Show that at least 3030 flight routes are needed to satisfy the two requirements.

7. Let a, b, c, and d be real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 13$$
$$a^2 + b^2 + c^2 + d^2 = 43.$$

Show that $ab \geq 3 + cd$.

8. In right triangle ABC, $\angle ACB = 90^\circ$ and $\tan A > \sqrt{2}$. M is the midpoint of AB, P is the foot of the altitude from C, and N is the midpoint of CP. Line AB meets the circumcircle of CNB again at Q. R lies on line BC such that QR and CP are parallel, S lies on ray CA past A such that $BR = RS$, and V lies on segment SP such that $AV = VP$. Line SP meets the circumcircle of CPB again at T. W lies on ray VA past A such that $2AW = ST$, and O is the circumcenter of SPM. Prove that lines OM and BW are perpendicular.