23rd Philippine Mathematical Olympiad

National Stage, Written Phase (Day 1)

19 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

1. In convex quadrilateral $ABCD$, $\angle CAB = \angle BCD$. P lies on line BC such that $AP = PC$, Q lies on line AP such that AC and DQ are parallel, R is the point of intersection of lines AB and CD, and S is the point of intersection of lines AC and QR. Line AD meets the circumcircle of AQS again at T. Prove that AB and QT are parallel.

2. Let n be a positive integer. Show that there exists a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\sum_{k=1}^{n} \frac{k}{(k + \sigma(k))^2} < \frac{1}{2}.$$

3. Denote by $\mathbb{Q}^+$ the set of positive rational numbers. A function $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfies

- $f(p) = 1$ for all primes p, and
- $f(ab) = af(b) + bf(a)$ for all $a, b \in \mathbb{Q}^+$.

For which positive integers n does the equation $nf(c) = c$ have at least one solution c in $\mathbb{Q}^+$?

4. Determine the set of all polynomials $P(x)$ with real coefficients such that the set $\{P(n) \mid n \in \mathbb{Z}\}$ contains all integers, except possibly finitely many of them.

23rd Philippine Mathematical Olympiad

National Stage, Written Phase (Day 2)

20 March 2021

Time: 4.5 hours *Each item is worth 7 points.*

5. A positive integer is called *lucky* if it is divisible by 7, and the sum of its digits is also divisible by 7. Fix a positive integer n. Show that there exists some lucky integer ℓ such that $|n - \ell| \leq 70$.

6. A certain country wishes to interconnect 2021 cities with flight routes, which are always two-way, in the following manner:

- There is a way to travel between any two cities either via a direct flight or via a sequence of connecting flights.
- For every pair (A, B) of cities that are connected by a direct flight, there is another city C such that (A, C) and (B, C) are connected by direct flights.

Show that at least 3030 flight routes are needed to satisfy the two requirements.

7. Let a, b, c, and d be real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 13$$
$$a^2 + b^2 + c^2 + d^2 = 43.$$

Show that $ab \geq 3 + cd$.

8. In right triangle ABC, $\angle ACB = 90^\circ$ and $\tan A > \sqrt{2}$. M is the midpoint of AB, P is the foot of the altitude from C, and N is the midpoint of CP. Line AB meets the circumcircle of CNB again at Q. R lies on line BC such that QR and CP are parallel, S lies on ray CA past A such that $BR = RS$, and V lies on segment SP such that $AV = VP$. Line SP meets the circumcircle of CPB again at T. W lies on ray VA past A such that $2AW = ST$, and O is the circumcenter of SPM. Prove that lines OM and BW are perpendicular.

23^{rd} Philippine Mathematical Olympiad

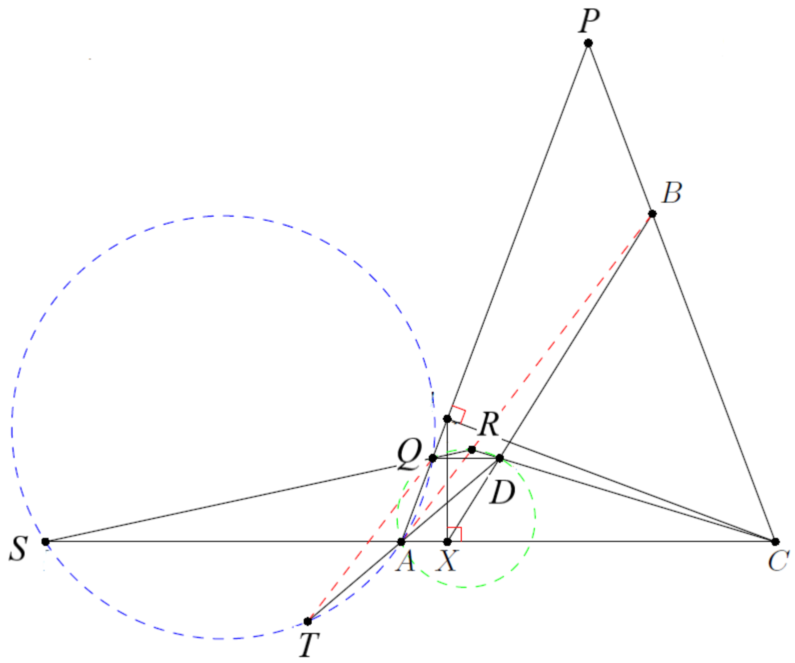
National Stage, Written Phase (Solutions)

19-20 March 2021

---

1. In convex quadrilateral $ABCD$, $\angle CAB = \angle BCD$. P lies on line BC such that $AP = PC$, Q lies on line AP such that AC and DQ are parallel, R is the point of intersection of lines AB and CD, and S is the point of intersection of lines AC and QR. Line AD meets the circumcircle of AQS again at T. Prove that AB and QT are parallel.

<u>*Solution.*</u> Refer to the figure shown below.



By angle-chasing (with directed angles), we have

$$\angle QAR = \angle PAB = \angle CAB - \angle CAP = \angle BCD - \angle PCA = \angle ACD = \angle QDR.$$

Thus quadrilateral $QADR$ is cyclic. Then,

$$\angle BAD = \angle RAD = \angle RQD = \angle QSA = \angle QTA = \angle QTD,$$

and hence AB is parallel to QT. $\square$

2. Let n be a positive integer. Show that there exists a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\sum_{k=1}^{n} \frac{k}{(k+\sigma(k))^2} < \frac{1}{2}.$$

Solution: It suffices to produce one such function. For this, consider the function $\sigma(k) = n + 1 - k$. Then note that σ is one-to-one, since for every a and b,

$$\sigma(a) = \sigma(b) \Rightarrow n + 1 - a = n + 1 - b \Rightarrow a = b.$$

In this case,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k}{(k+\sigma(k))^2} &= \sum_{k=1}^{n} \frac{k}{(k+n+1-k)^2} \\
&= \sum_{k=1}^{n} \frac{k}{(n+1)^2} \\
&= \frac{1}{(n+1)^2} \cdot \sum_{k=1}^{n} k \\
&= \frac{1}{2} \cdot \frac{n}{n+1} \\
&< \frac{1}{2}.
\end{aligned}$$

□

3. Denote by $\mathbb{Q}^+$ the set of positive rational numbers. A function $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfies

- $f(p) = 1$ for all primes p, and
- $f(ab) = af(b) + bf(a)$ for all $a, b \in \mathbb{Q}^+$.

For which positive integers n does the equation $nf(c) = c$ have at least one solution c in $\mathbb{Q}^+$?

Solution: We claim that either n is the product of distinct primes, or $n = 1$. Define $g(x) = \frac{f(x)}{x}$. The equation we are trying to solve becomes $g(c) = \frac{1}{n}$. The definition of the function becomes $g(p) = \frac{1}{p}$ for all primes p, and

$$g(ab) = g(a) + g(b). \tag{1}$$

Substituting $a = 1$ in the above yields $g(1) = 0$. Letting $b = \frac{1}{a}$ in (1) and using $g(1) = 0$ gives $g(a) = -g(\frac{1}{a})$ for all a in $\mathbb{Q}^+$. An easy induction then proves that

$$g(a^n) = ng(a) \tag{2}$$

for positive integers n and a in $\mathbb{Q}^+$. This gives

$$g(p^e) = \frac{e}{p} \tag{3}$$

for prime p and positive integers e. These facts, combined, give us the general formula for $g(\frac{p}{q})$ in terms of the prime factorizations of p and q:

$$g\left(\frac{p_1^{e_1}\cdots p_k^{e_k}}{q_1^{f_1}\cdots q_\ell^{f_\ell}}\right) = \frac{e_1}{p_1} + \cdots + \frac{e_k}{p_k} - \frac{f_1}{q_1} - \cdots - \frac{f_\ell}{q_\ell} = \frac{m}{p_1\cdots p_k q_1 \cdots q_\ell},$$

for some integer m. Observe that the denominator is a product of distinct primes. Thus, if $g(c) = \frac{1}{n}$ for some c, then either n is the product of distinct primes, or $n = 1$. It remains to prove that all such n have such a solution c.

When $n = 1$, taking $c = p^p$ for some prime p works by (3). We now prove that if $g(c) = \frac{1}{n}$ for some $c \in \mathbb{Q}^+$ and positive integer n, then there exists $d \in \mathbb{Q}^+$ such that $g(d) = \frac{1}{np}$, for any prime p relatively prime to n. This finishes the problem by induction.

Let x and y be integers whose values will be determined later. Observe that, by (2), $g(c^x) = xg(c)$. By (3), we get $g(p^y) = \frac{y}{p}$. Finally, using (1) on c^x and p^y gives

$$g(c^x p^y) = \frac{x}{n} + \frac{y}{p} = \frac{px + ny}{np}.$$

It remains to choose integers x and y such that $px + ny = 1$. But by Bézout's identity, as the greatest common divisor of p and n is 1, there do exist such integers. Taking $d = c^x p^y$ then gives $g(d) = \frac{1}{np}$, finishing the problem. $\square$

4. Determine the set of all polynomials $P(x)$ with real coefficients such that the set $\{P(n) \mid n \in \mathbb{Z}\}$ contains all integers, except possibly finitely many of them.

Solution. We claim that the only such polynomials are of the form $P(x) = \frac{1}{i}(x + j)$ for some integers $i \neq 0, j$.

Let $\mathcal{R}$ be the set $\{P(n) \mid n \in \mathbb{Z}\}$.

Without loss of generality, we may assume that the leading coefficient of $P(x)$ is positive; otherwise we can consider $-P(x)$. If the polynomial $P(x)$ has even degree, then it must have a minimum value m. Then all integers less than m are in the set $\mathbb{Z}\setminus\mathcal{R}$, so it cannot be finite. Thus $P(x)$ must have odd degree.

Since the set $\mathbb{Z}\setminus\mathcal{R}$ is finite, there exists some M_1 such that $x \in \mathcal{R}$ for all integers $x > M_1$. As $P(x)$ has odd degree and positive leading coefficient, there exists some M_2 such that $P(M_2) > M_1$, $P(x) \leq P(M_2)$ for all $x \leq M_2$ and $P(x)$ is increasing on $[M_2, \infty)$. Let $M = \max\{M_1, M_2\}$.

Choose an integer n such that $n > M$ and $P(n+1) > P(n)+1$. Letting $s = \lfloor P(n)+1 \rfloor$, we claim there is no such integer x such that $P(x) = s$. Consider the following cases:

- $x \leq M_2$. Then $P(x) \leq P(M_2) \leq P(n) < s$.
- $M_2 < x \leq n$. Then $P(x) \leq P(n) < s$ as $P(x)$ is increasing.
- $x > n$. Then $x \geq n+1$ and $P(x) \geq P(n+1) > P(n)+1 \geq s$, as $P(x)$ is increasing.

Thus s is in the set $\mathbb{Z} \setminus \mathcal{R}$. However, $s > P(n) > P(M_2) > M_1$, contradicting the definition of M_1. Thus, $P(n+1) \leq P(n) + 1$ for all integers $n > M$.

Let d be the degree of $P(x)$. Observe $P(n+1) - P(n)$ is a polynomial of degree $d - 1$ with the same positive leading coefficient as $P(x)$. If $d - 1 \geq 1$, then $P(n+1) - P(n)$ will become arbitrarily large as n increases, contradicting $P(n+1) \leq P(n) + 1$ for all integers $n > M$.

Therefore, $d = 1$, and $P(x) = ax + b$ for some real numbers $a \neq 0$ and b. As $\mathbb{Z} \setminus \mathcal{R}$ is finite, only finitely many pairs of integers $(t, t+1)$ are not in $\mathcal{R}$. Thus, there exists distinct integers n_1 and n_2 such that $P(n_1) = t$ and $P(n_2) = t + 1$. It follows that

$$1 = P(n_2) - P(n_1) = a(n_2 - n_1) \implies a = \frac{1}{n_2 - n_1} = \frac{1}{i}$$

for some integer $i \neq 0$. Furthermore,

$$P(n_1) = t = an_1 + b \implies b = \frac{it - n_1}{i} = \frac{j}{i}$$

for some integer j. Hence $P(x)$ must be of the form $\frac{1}{i}(x + j)$ for integers $i \neq 0, j$. All such polynomials clearly satisfy the given conditions. □

5. A positive integer is called *lucky* if it is divisible by 7, and the sum of its digits is also divisible by 7. Fix a positive integer n. Show that there exists some lucky integer ℓ such that $|n - \ell| \leq 70$.

Solution. Suppose we have some lucky integer n. We will show that the gap between it and the next lucky integer is no more than 2×70.

In one iteration, we increment $n \to n + 7$. Clearly the number we have is still divisible by 7, so it will suffice for us to show that the digit-sum will be divisible by 7 after some number of iterations.

First, suppose that the last two digits of n are less than 30. We claim that after at most 10 such iterations, the digit-sum cycles through all possible values mod 7.

Note that if the last digit is less than 3, then the digit-sum mod 7 does not change, while if the last digit is 3 or more, then the digit-sum mod 7 decreases by 2. This follows because $10 \equiv -3 \bmod 7$, but we do still add $+1$ for the carrying in the tens place. Since we assumed that the last two digits are less than 30, there will never be any carrying in the hundreds place onwards within our 10 iterations (since $d < 30$ implies that $d + 70 < 100$).

Now, note that 7 is coprime to 10. So, our 10 iterations of $+7$ will have the last digit cycle through all the digits from 0 to 9 exactly once each. Thus, the cases which do not change the digit-sum mod 7 (which are $0 \to 7$, $1 \to 8$, and $2 \to 9$) happen exactly once each, and the other seven iterations do -2 on the digit-sum.

Because -2 is coprime to 7, 7 iterations of -2 will have the digit-sum cycle through all values mod 7. So, we are guaranteed to be able to achieve a digit-sum of $0 \pmod 7$ within our 10 iterations of $+7$.

If the last two digits are 30 or more, we can make the last two digits less than 30 with at most 10 iterations of $+7$ (because if $30 \leq d < 100$, then $100 - d \leq 70$).

Thus, from a lucky integer, we can product the next lucky integer in 20 iterations or less. The gap between lucky integers is at most 140, and so any integer is at most 70 away from a lucky integer. □

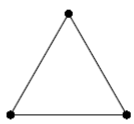
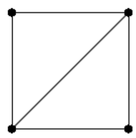
6. A certain country wishes to interconnect 2021 cities with flight routes, which are always two-way, in the following manner:

- There is a way to travel between any two cities either via a direct flight or via a sequence of connecting flights.
- For every pair (A, B) of cities that are connected by a direct flight, there is another city C such that (A, C) and (B, C) are connected by direct flights.

Show that at least 3030 flight routes are needed to satisfy the two requirements.

<u>*Solution.*</u> More generally, consider the graph G with n vertices representing n cities, two of them being connected by an edge if there is a two-way flight between them. We will prove that if G is connected and every edge in G belongs to a triangle, then G must have at least $\lfloor (3n-2)/2 \rfloor$ edges. We call such a graph a *T-graph*. Let a_n be the minimum number of edges of a T-graph with n vertices and $b_n = \lfloor (3n-2)/2 \rfloor$.

- We first show that $a_n \leq b_n$ for $n \geq 3$ via an explicit construction. We clearly have $a_3 = b_3 = 3$. If $n = 2m+1$ is odd, we form m triangles having one common vertex. This is a T-graph with $3m$ edges, so $a_n \leq 3m = b_n$. If $n = 2m$ is even, we form the above T-graph with $2m-1$ vertices, choose an edge, say XY, and then add a vertex Z and edges XZ and YZ. This results in a T-graph with n vertices and $3(m-1)+2 = 3m-1$ edges, so $a_n \leq 3m-1 = b_n$.
- We next show that $a_n \geq b_n$ for $n \geq 3$, that is, any T-graph with n vertices must have at least b_n edges. We proceed by strong induction. The cases $n = 3$ and $n = 4$ can be easily verified, as shown by their respective T-graphs below.

We claim that in a T-graph G with n vertices and less than b_n edges, there is a vertex whose degree is 2. Suppose otherwise; since G is connected, every vertex has degree at least 1 and since every edge in G belongs to a triangle, every vertex must have degree at least 2. If each vertex has degree at least 3, then the sum of the degrees is at least $3n$. But that sum is twice the number of edges, which is less than $2b_n < 3n$, a contradiction. We now suppose $a_k = b_k$ for all $k < n$[1]; we will show that $a_n \geq b_n$. Consider a T-graph G with n vertices and a_n edges such that $a_n < b_n$. Then by assumption and our claim above, there is a vertex v of G of degree 2 which must be in some triangle, say uvw. Suppose the edge uw belongs to some other triangle. Then the graph H obtained by deleting v and edges uv and vw is a T-graph with $n-1$ vertices and $a_n - 2 < b_n - 2 \leq b_{n-1}$, a contradiction. Thus, uw does not lie on any triangle other than uvw. We now construct a smaller graph G' as follows: we delete v and *contract* uw, that is, we collapse the edge uw by combining its endpoints into a single vertex, say x. In G', we connect a vertex y to x if in G, y is adjacent to one of u or w. We observe that G' is a T-graph with $n-2$ vertices and has 3 fewer edges than G (as we remove uv, uw and vw). We see that G' cannot have at least 4 fewer edges than G; otherwise some vertex y in G would

[1]That is, for every $k < n$, any T-graph with k vertices has at least $a_k = b_k$ edges.

be adjacent to both u and w, thus forming a triangle uyw in G, a contradiction. Thus, G' has $a_n - 3 < b_n - 3 \leq b_{n-2}$ edges, contradicting the induction hypothesis. Hence, we must have $a_n \geq b_n$.

Combining these results leads us to $a_n = b_n$, and setting $n = 2021$ yields the desired answer. □

7. Let a, b, c, d be real numbers such that $a \geq b \geq c \geq d$ and which satisfy the system of equations

$$\begin{aligned} a+b+c+d &= 13 \quad (4)\\ a^2+b^2+c^2+d^2 &= 43 \quad (5) \end{aligned}$$

Show that $ab \geq 3 + cd$.

Solution. Since $(a-d)(b-c) \geq 0$ and $(a-b)(c-d) \geq 0$, then

$$ab + cd \geq ac + bd \geq ad + bc \qquad (6)$$

From Equations (4) and (5), we have

$$(ab+cd)+(ac+bd)+(ad+bc) = \frac{1}{2}\left[(a+b+c+d)^2 - (a^2+b^2+c^2+d^2)\right] = \frac{1}{2}\left(13^2 - 43\right) = 63$$

Thus, using Equation (6), we have $ab + cd \geq \dfrac{63}{3} = 21$. Since $c + d = 13 - (a+b)$, then

$$\begin{aligned} (a+b)^2 + [13-(a+b)]^2 &= (a+b)^2 + (c+d)^2 \\ &= (a^2+b^2+c^2+d^2) + 2(ab+cd) \\ &\geq 43 + 2(21) = 85 \end{aligned}$$

Thus, we have

$$\begin{aligned} (a+b)^2 + 169 - 26(a+b) + (a+b)^2 &\geq 85 \\ (a+b)^2 - 13(a+b) + 42 &\geq 0 \\ (a+b-6)(a+b-7) &\geq 0 \end{aligned}$$

which means either $a+b \leq 6$ or $a+b \geq 7$. However, since $2(a+b) \geq a+b+c+d = 13$, then $a+b \geq 6.5$. Thus, $a+b \geq 7$. From this, we have

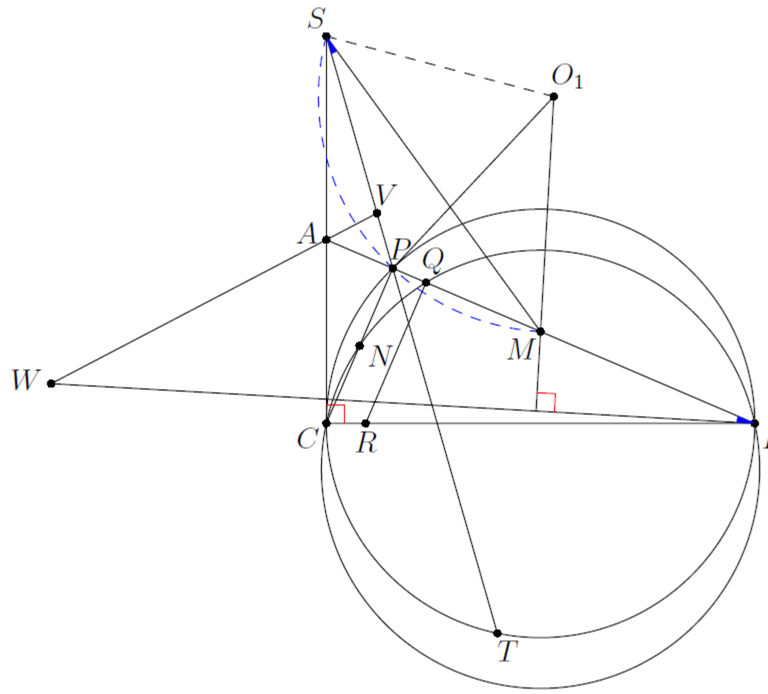
$$\begin{aligned} (a+b)^2 + (c-d)^2 &\geq 7^2 + 0^2 \\ a^2+b^2+c^2+d^2+2ab-2cd &\geq 49 \\ 43 + 2(ab-cd) &\geq 49 \\ ab - cd &\geq 3 \\ ab &\geq 3 + cd \end{aligned}$$

□

(*Note:* This problem is a modified version of Problem A3 of the 2005 IMO Shortlist.)

8. In right triangle ABC, $\angle ACB = 90^\circ$ and $\tan A > \sqrt{2}$. M is the midpoint of AB, P is the foot of the altitude from C, and N is the midpoint of CP. Line AB meets the circumcircle of CNB again at Q. R lies on line BC such that QR and CP are parallel, S lies on ray CA past A such that $BR = RS$, and V lies on segment SP such that $AV = VP$. Line SP meets the circumcircle of CPB again at T. W lies on ray VA past A such that $2AW = ST$, and O is the circumcenter of SPM. Prove that lines OM and BW are perpendicular.

Solution. Refer to the figure shown below.



Since CP is the C-altitude of ABC, triangles ACP, ABC and CBP are similar, so $AP = \frac{AC^2}{AB}$ and $BP = \frac{BC^2}{AB}$. As $CNQB$ is cyclic, we have $\angle PNQ = \angle CBP = \angle ACP$ so triangles ACP and QNP are similar. With N being the midpoint of PC, we have $\frac{AP}{PQ} = \frac{PC}{PN} = 2$, which gives $PQ = \frac{1}{2}AP$ and $BQ = AP - AQ = AB - \frac{3}{2}AP$. We now claim that $SC^2 = 2AB \cdot PM$. Indeed, since QR and CP are parallel, we have $\frac{BQ}{BP} = \frac{BR}{BC}$ by Thales' Theorem. Thus,

$$RS = BR = \frac{BC \cdot BQ}{BP} = \frac{BC(AB - \frac{3AC^2}{2AB})}{\frac{BC^2}{AB}} = \frac{2AB^2 - 3AC^2}{2BC}.$$

Applying Pythagorean theorem on triangle SCR, we see that

$$\begin{aligned} SC^2 &= BR^2 - RC^2 = BC(2BR - BC) = BC\left(\frac{2AB^2 - 3AC^2}{BC} - BC\right) \\ &= 2BC^2 + 2CA^2 - 3CA^2 - BC^2 = BC^2 - CA^2. \end{aligned}$$

Note that $\tan A > \sqrt{2}$ implies that S is indeed on ray CA past A. On the other hand, with M being the midpoint of AB, we compute

$$PM = AM - AP = \frac{AB}{2} - \frac{CA^2}{AB} = \frac{AB^2 - 2CA^2}{2AB} = \frac{BC^2 - CA^2}{2AB} = \frac{SC^2}{2AB}$$

which proves the desired claim.

Observe that the circumcircle of CPB is tangent to SC at C, so the Power of the Point gives $SC^2 = SP \cdot ST$. It follows from the above claim that

$$SP \cdot ST = 2AB \cdot PM \iff \frac{SP}{PM} = \frac{2AB}{ST} = \frac{2AB}{2AW} = \frac{AB}{AW}.$$

Since $AV = PV$, we have $\angle WAB = 180^\circ - \angle VAP = 180^\circ - \angle VPA = \angle MPS$, so triangles WAB and MPS are similar. Thus, we see that $\angle PSM = \angle ABW$ and with $OP = OM$, we arrive at

$$\angle ABW + \angle PMO = \angle PSM + \angle PMO = \frac{1}{2}(\angle POM + 2\angle PMO) = \frac{1}{2} \cdot 180^\circ = 90^\circ.$$

Hence, we get $OM \perp BW$, which completes the proof. $\square$