24$^{\text{th}}$ Philippine Mathematical Olympiad

National Stage (Day 1)

18 March 2022

Time: 4.5 hours *Each item is worth 7 points.*

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers a, b, c, and d.

2. The PMO Magician has a special party game. There are n chairs, labelled 1 to n. There are n sheets of paper, labelled 1 to n.

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks n party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if $1 < m \leq n$, where m is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly m claps.

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer k, denote by S_k the set of all visible lattice points (x, y) such that $x^2 + y^2 = k^2$. Let D denote the set of all positive divisors of $2021 \cdot 2025$. Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point (x, y) on the plane where both x and y are integers, and $|A|$ denotes the number of elements of the set A.

4. Let $\triangle ABC$ have incenter I and centroid G. Suppose that P_A is the foot of the perpendicular from C to the exterior angle bisector of B, and Q_A is the foot of the perpendicular from B to the exterior angle bisector of C. Define P_B, P_C, Q_B, and Q_C similarly. Show that P_A, P_B, P_C, Q_A, Q_B, and Q_C lie on a circle whose center is on line IG.

24^{th} Philippine Mathematical Olympiad

National Stage (Day 2)

19 March 2022

Time: 4.5 hours *Each item is worth 7 points.*

5. Find all positive integers n for which there exists a set of exactly n distinct positive integers, none of which exceed n^2, whose reciprocals add up to 1.

6. In $\triangle ABC$, let D be the point on side BC such that $AB+BD = DC+CA$. The line AD intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. Prove that one of the common tangents of the circumcircles of $\triangle BDX$ and $\triangle CDX$ is parallel to BC.

7. Let a, b, and c be positive real numbers such that $ab + bc + ca = 3$. Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

8. The set $S = \{1, 2, \ldots, 2022\}$ is to be partitioned into n disjoint subsets $S_1, S_2, \ldots, S_n$ such that for each $i \in \{1, 2, \ldots, n\}$, exactly one of the following statements is true:

(a) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) > 1$.

(b) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) = 1$.

Find the smallest value of n for which this is possible.

24^{th} Philippine Mathematical Olympiad

National Stage (Solutions)

18-19 March 2022

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1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \le (a-c)f(b-d)$$

for all real numbers a, b, c, and d.

<u>Solution.</u> We claim that the only solutions are $f(x) = 0$ and $f(x) = x$. It is easy to see that both satisfy the functional inequality. We now show that no other functions satisfy the given condition.

Letting $a = b = c = d$ yields $2f(0)^2 \le 0$ which implies $f(0) = 0$. Now, suppose that $f \not\equiv 0$. Then we can find $p \in \mathbb{R}$ such that $f(p) \neq 0$. For $a \in \mathbb{R}$, let $b = a$, $c = 0$, and $d = a - p$. We then have

$$f(p)f(a) \le af(p).$$

Case 1: Suppose $f(p) > 0$. Then $f(a) \le a$ for all $a \in \mathbb{R}$. In particular $f(-1) < 0$. In the original functional equation, letting $b = a$, $c = 0$, and $d = a + 1$ yields $f(-1)f(a) \le af(-1)$ for all $a \in \mathbb{R}$. Hence, $f(a) \ge a$ for all $a \in \mathbb{R}$. Thus for this case, $f(x) = x$ for all $x \in \mathbb{R}$.

Case 2: Suppose $f(p) < 0$. Then $f(a) \ge a$ for all $a \in \mathbb{R}$. In particular $f(1) > 0$. In the original functional equation, letting $b = a$, $c = 0$, and $d = a - 1$ yields $f(1)f(a) \le af(1)$ for all $a \in \mathbb{R}$. Hence, $f(a) \le a$ for all $a \in \mathbb{R}$. Thus for this case, $f(x) = x$ for all $x \in \mathbb{R}$.

□

2. The PMO Magician has a special party game. There are n chairs, labelled 1 to n. There are n sheets of paper, labelled 1 to n.

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks n party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if $1 < m \le n$, where m is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly m claps.

<u>Solution.</u> Decompose the permutation into cycles of lengths $c_1, c_2, c_3, \ldots, c_k$. Note that $c_1 + c_2 + \cdots + c_k = n$. A guest in a cycle of length c returns to their original seat after c claps. Thus, all guests return to their original seats after $\operatorname{lcm}\{c_i\}$ claps.

Let $m = pq$, where p and q are coprime and both greater than 1; if $m > 1$ is not a prime power, then it has at least two different prime factors, so this is always possible. Suppose we have at least one cycle of lengths p and q, and *all* cycles are of lengths p or q. Then, all guests will return to their original seats after $\operatorname{lcm}\{p, q\} = m$ claps.

If $m = n$, we can just connect all n party guests in one big cycle of size $m = n$. Otherwise, $m < n$, and note that the problem is equivalent to finding a nonnegative integer solution to $p\overline{x} + q\overline{y} = n - p - q$. Since $m < n$, a solution always exists by the Chicken McNugget Theorem, completing the proof. □

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer k, denote by S_k the set of all visible lattice points (x, y) such that $x^2 + y^2 = k^2$. Let D denote the set of all positive divisors of $2021 \cdot 2025$. Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point (x, y) on the plane where both x and y are integers, and $|A|$ denotes the number of elements of the set A.

<u>Solution.</u> We claim that the required sum is 20.

Let T_k denote the set of all lattice points in the circle $x^2 + y^2 = k^2$. We claim that $\sum_{d|k} |S_d| = |T_k|$. Indeed, given a point (x, y) in T_k, let $g = \gcd(x, y)$. Then $x/g, y/g$ are necessarily coprime, and hence $(x/g, y/g)$ visible, and $(x/g)^2 + (y/g)^2 = (k/g)^2$. This implies $(x/g, y/g) \in \bigcup_{d|k} S_d$. Next, note that the S_d's are necessarily disjoint. Now if (x', y') is a visible lattice point in S_d where $d|k$, then we can write $k = gd$ so that $(x, y) = (gx', gy')$ is a lattice point in T_k. This establishes a bijection between $\bigcup_{d|k} S_d$ and T_k, and since the S_d's are disjoint, the claim follows.

From the claim, it suffices to find the number of lattice points in the circle $x^2 + y^2 = (2021 \cdot 2025)^2$. This is equivalent to

$$x^2 + y^2 = 3^8 \cdot 5^4 \cdot 43^2 \cdot 47^2.$$

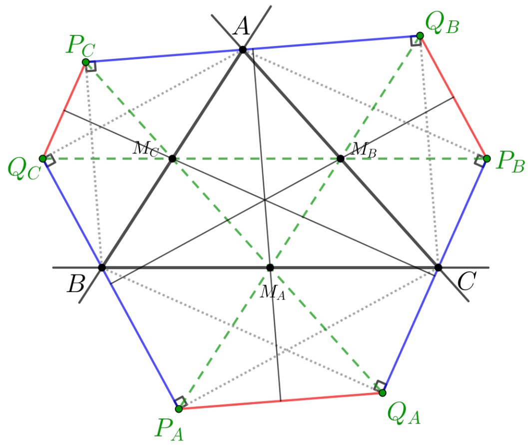
Now it is well-known that if $x^2 + y^2 \equiv 0 \pmod{p}$ where $p \equiv 3 \pmod{4}$ is a prime, then $x \equiv y \equiv 0 \pmod{p}$. Thus, we must also have $x, y \equiv 0 \pmod{3^4 \cdot 43 \cdot 47}$. It then follows that the number of lattice points is the same as the number of lattice points in $x^2 + y^2 = 25^2$.

If $x = 0$ or $y = 0$, there are 4 solutions. Otherwise, assume WLOG that they are both positive. Now it is well-known that all solutions to $x^2 + y^2 = z^2$ are in the form $x = g(m^2 - n^2)$, $y = 2gmn$, and $z = g(m^2 + n^2)$, where $m > n$ are coprime positive integers, and g is a positive integer. Thus, we want $g(m^2 + n^2) = 25$. Note that $g|25$, so $g = 1, 5, 25$.

If $g = 25$, then $m^2 + n^2 = 1$, so $n = 0$, contradiction. If $g = 5$, then $m^2 + n^2 = 5$, which yields $m = 2$ and $n = 1$ and thus $g(m^2 - n^2) = 15$ and $2gmn = 20$, so $(x, y) = (15, 20), (20, 15)$. If $g = 1$, then we get $m^2 + n^2 = 25$, from which we obtain $m = 4$ and $n = 3$. It then follows that $(x, y) = (24, 7), (7, 24)$, and so we have 2 solutions when x, y are both positive. This implies that there are $4 \cdot 4 = 16$ solutions when x, y are nonzero, and so there are $4 + 16 = 20$ solutions in total. □

4. Let $\triangle ABC$ have incenter I and centroid G. Suppose that P_A is the foot of the perpendicular from C to the exterior angle bisector of B, and Q_A is the foot of the perpendicular from B to the exterior angle bisector of C. Define P_B, P_C, Q_B, and Q_C similarly. Show that P_A, P_B, P_C, Q_A, Q_B, and Q_C lie on a circle whose center is on line IG.

<u>Solution.</u> Refer to the figure shown below:



Let $M_A, M_B,$ and M_C be the midpoints of $BC, CA,$ and AB respectively.

First, it may be shown that P_B and Q_C lie on M_BM_C.

Note that $\angle AM_CQ_C = 2\angle ABQ_C = 180° - \angle ABC = \angle BM_CM_B$. Thus, Q_C lies on M_BM_C. Likewise, $AM_BP_B = 2\angle ACP_B = 180° - \angle ACB = \angle CM_BM_C$. Thus, P_B also lies on M_BM_C.

Similarly, P_C and Q_A lie on M_CM_A, and P_A and Q_B lie on M_AM_B.

Now, as M_A is the center of the circle passing through $B, C, Q_A,$ and P_A, then $M_AP_A = M_AQ_A$, so the angle bisector of $\angle M_CM_AM_B$ coincides with the perpendicular bisector of P_AQ_A.

Observe that $M_CQ_A = M_CM_A + M_AQ_A = \frac{CA}{2} + \frac{BC}{2} = M_BP_B + M_CM_B = M_CP_B$. Thus, the perpendicular bisector of Q_AP_B coincides with the angle bisector of $\angle M_BM_CM_A$.

Using similar observations, it may then be concluded that the perpendicular bisectors of P_AQ_A, Q_AP_B, P_BQ_B, Q_BP_C, P_CQ_C, and Q_CP_A all concur at the incenter of $\triangle M_AM_BM_C$. Thus, the latter must also be the center of the circle containing all six points. As the medial triangle is the image of a homothety on $\triangle ABC$ with center G having a scale factor of -0.5, then the incenter of $\triangle M_AM_BM_C$ must lie on IG. □

5. Find all positive integers n for which there exists a set of exactly n distinct positive integers, none of which exceed n^2, whose reciprocals add up to 1.

<u>Solution.</u> The answer is all $n \neq 2$. For $n = 1$, the set $\{1\}$ works. For $n = 2$, no set exists, simply because the sum of reciprocals of two distinct integers cannot be equal to 1. For $n = 3$, take $\{2, 3, 6\}$.

For $n > 3$, the identity

$$\frac{1}{k} = \frac{1}{k+r} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(k+r-1)(k+r)}$$

allows us to extend a sum of t terms to one of exactly $t+r$ terms. Taking $k=3$ and $r=n-3$ allows us to turn the sum $1=1/2+1/3+1/6$ to the n-term sum

$$1=\frac{1}{2}+\frac{1}{6}+\frac{1}{n}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-1)n}.$$

This construction works provided that $n\neq k(k+1)$ for any k. Otherwise, we have $n\geq 6$, and instead we apply the above to the sum $1=1/2+1/3+1/10+1/15$, taking $k=3$, $r=n-4$ to yield

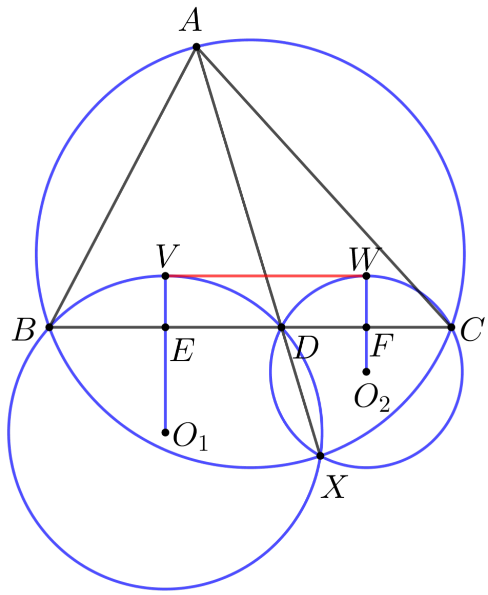
$$1=\frac{1}{2}+\frac{1}{10}+\frac{1}{15}+\frac{1}{n-1}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-2)(n-1)}.$$

The above then works, because if $n=k(k+1)$ for some k, then $n-1\neq 2,10,15$, and $n-1\neq m(m+1)$ for any m by parity, since $n-1$ is odd and $m(m+1)$ is always even.

This construction is not unique; there are other similar ones. □

6. In $\triangle ABC$, let D be the point on side BC such that $AB+BD=DC+CA$. The line AD intersects the circumcircle of $\triangle ABC$ again at point $X\neq A$. Prove that one of the common tangents of the circumcircles of $\triangle BDX$ and $\triangle CDX$ is parallel to BC.

Solution. Refer to the figure shown below:



Let V and W be the midpoints of arcs BD and CD respectively. We claim that VW is the desired common tangent. To prove this, let E and F be the orthogonal projections of V and W onto BC. Note that E and F are the midpoints of BD and CD respectively. Now we claim that $VE=WF$. To see this, note that

$$\begin{aligned}
VE &= BV\sin\angle VBE\\
&= \frac{BX\sin\angle BXV}{\sin\angle BDX}\sin\frac{\angle BXD}{2}\\
&= \frac{BX\sin\frac{\angle BXA}{2}}{\sin\angle BDX}\sin\frac{C}{2}\\
&= \frac{BX}{\sin\angle BDX}\sin^2\frac{C}{2}.
\end{aligned}$$

Similarly, we can prove that $WF = \frac{CX}{\sin\angle CDX}\sin^2\frac{B}{2}$. Thus, to prove the claim, it suffices to prove that $\dfrac{BX}{CX} = \dfrac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}$. This is because

$$\frac{BX}{CX} = \frac{\sin\angle BAD}{\sin\angle CAD} = \frac{c/BD}{b/CD} = \frac{c(s-c)}{b(s-b)} = \frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}.$$

This proves the first claim.

Next, we claim that VW is the desired common tangent. Note that from the first claim, $VWFE$ is a rectangle, since $\angle VEF$ and $\angle WFE$ are both right angles. Let O_1 and O_2 be the circumcenters of triangles BDX and CDX respectively. Then $O_1V \perp EF$, so since $EF||VW$ we get $O_1V \perp VW$. Likewise, $O_2W \perp VW$ as well, which proves the second claim.

It then follows that VW is the desired common tangent parallel to BC, and the required conclusion follows. □

7. Let a, b, and c be positive real numbers such that $ab + bc + ca = 3$. Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

<u>Solution.</u> It can be shown that

$$\frac{1}{1+a^4} \geq \frac{2-a^2}{2}.$$

Indeed, simplifying yields

$$\begin{aligned} 2 &\geq (1+a^4)(2-a^2) \\ (a^4+1)(a^2-2)+2 &\geq 0 \\ a^6-2a^4+a^2-2+2 &\geq 0 \\ a^6-2a^4+a^2 &\geq 0 \\ a^2(a^4-2a^2+1) &\geq 0 \\ a^2(a^2-1)^2 &\geq 0 \end{aligned}$$

which is true.

Thus

$$\begin{aligned} \frac{1}{1+a^4} &\geq \frac{2-a^2}{2} \\ \frac{bc}{1+a^4} &\geq \frac{2bc-a^2bc}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \frac{ca}{1+b^4} &\geq \frac{2ca-ab^2c}{2}, \text{ and} \\ \frac{ab}{1+c^4} &\geq \frac{2ab-abc^2}{2}. \end{aligned}$$

Adding these up yields

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq (bc+ca+ab) - \frac{a^2bc+ab^2c+abc^2}{2}.$$

Observe that

$$\begin{aligned}(ab+bc+ca)^2 &\geq 3((ab)(bc)+(bc)(ca)+(ca)(ab))\\ 9 &\geq 3(a^2bc+ab^2c+abc^2)\\ 3 &\geq a^2bc+ab^2c+abc^2.\end{aligned}$$

Thus,

$$(bc+ca+ab) - \frac{a^2bc+ab^2c+abc^2}{2} \geq 3 - \frac{3}{2} = \frac{3}{2}.$$

Therefore,

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

□

8. The set $S = \{1, 2, \ldots, 2022\}$ is to be partitioned into n disjoint subsets $S_1, S_2, \ldots, S_n$ such that for each $i \in \{1, 2, \ldots, n\}$, exactly one of the following statements is true:

(a) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) > 1$.

(b) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) = 1$.

Find the smallest value of n for which this is possible.

Solution. The answer is 15.

Note that there are 14 primes at most $\sqrt{2022}$, starting with 2 and ending with 43. Thus, the following partition works for 15 sets. Let $S_1 = \{2, 4, \ldots, 2022\}$, the multiples of 2 in S. Let $S_2 = \{3, 9, 15 \ldots, 2019\}$, the remaining multiples of 3 in S not in S_1. Let $S_3 = \{5, 25, 35, \ldots, 2015\}$ the remaining multiples of 5, and so on and so forth, until we get to $S_{14} = \{43, 1849, 2021\}$. S_{15} consists of the remaining elements, i.e. 1 and those numbers with no prime factors at most 43, i.e., the primes greater than 43 but less than 2022: $S_{15} = \{1, 47, 53, 59, \ldots, 2017\}$. Each of $S_1, S_2, \ldots, S_{14}$ satisfies i., while S_{15} satisfies ii.

We show now that no partition in 14 subsets is possible. Let a Type 1 subset of S be a subset S_i for which i. is true and there exists an integer $d > 1$ for which d divides every element of S_i. Let a Type 2 subset of S be a subset S_i for which ii. is true. Finally, let a Type 3 subset of S be a subset S_i for which i. is true that is not a Type 1 subset. An example of a Type 3 subset would be a set of the form $\{pq, qr, pr\}$ where p, q, r are distinct primes.

Claim: Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of prime numbers, where p_k is the kth prime. Every optimal partition of the set $S(k) := \{1, 2, \ldots, p_k^2\}$, i.e., a partition with the least possible number of subsets, has at least $k-1$ Type 1 subsets. In particular, every optimal partition of this set has $k+1$ subsets in total. To see how this follows, we look at two cases:

- If every prime $p \le p_k$ has a corresponding Type 1 subset containing its multiples, then a similar partitioning to the above works: Take S_1 to S_k as Type 1 subsets for each prime, and take S_{k+1} to be everything left over. S_{k+1} will never be empty, as it has 1 in it. While in fact it is known that, for example, by Bertrand's postulate there is always some prime between p_k and p_k^2 so S_{k+1} has at least two elements, there is no need to go this far–if there were no other primes you could just move 2 from S_1 into S_{k+1}, and if $k > 1$ then S_1 will still have at least three elements remaining. And if $k = 1$, there is no need to worry about this, because $2 < 3 < 2^2$.
- On the other hand, if $p \le p_k$ has no corresponding Type 1 subset, then p and p^2 will not be contained in a Type 1 set. Neither can p nor p^2 be contained in a Type 3 set. If $\gcd(p, x) > 1$ for all x in the same set as p, then $\gcd(p, x) = p$, which implies that p is in a Type 1 set with $d = p$. Similarly, if $\gcd(p^2, x) > 1$ for all x in the same set as p^2, then $p \mid \gcd(p^2, x)$ for all x, and so p^2 is in a Type 1 set with $d = p$ as well. Hence p and p^2 must in fact be in Type 2 sets, and they cannot be in the same Type 2 set (as they share a common factor of $p > 1$); this means that the optimal partition has at least $k + 1$ subsets in total. A possible equality scenario for example is the sets $S_1 = \{1, 2, 3, 5, \ldots, p_k\}$, $S_2 = \{4, 9, 25, \ldots, p_k^2\}$, and S_3 to S_{k+1} Type 1 sets taking all remaining multiples of $2, 3, 5, \ldots, p_{k-1}$. This works, as p_k and p_k^2 are the only multiples of p_k in $S(k)$ with no prime factor other than p_k and thus cannot be classified into some other Type 1 set.

To prove our claim: We proceed by induction on k. Trivially, this is true for $k = 1$. Suppose now that any optimal partition of the set $S(k)$ has at least $k-1$ Type 1 subsets, and thus at least $k + 1$ subsets in total. Consider now a partition of the set $S(k + 1)$, and suppose that this partition would have at most $k + 1$ subsets. From the above, there exist at least two primes p, q with $p < q \le p_{k+1}$ for which there are no Type 1 subsets. If $q < p_{k+1}$ we have a contradiction. Any such partition can be restricted to an optimal partition of $S(k)$ with $p < q \le p_k$ having no corresponding Type 1 subsets. This contradicts our inductive hypothesis. On the other hand, suppose that $q = p_{k+1}$. Again restricting to $S(k)$ gives us an optimal partition of $S(k)$ with at most $k - 1$ Type 1 sets; the inductive hypothesis tells us that this partition has in fact exactly $k-1$ Type 1 sets and two Type 2 sets from a previous argument establishing the consequence of the claim. However, consider now the element pq. This cannot belong in any Type 1 set, neither can it belong in the same Type 2 set as p or p^2. Thus in addition to the given $k - 1$ Type 1 sets and 2 Type 2 sets, we need an extra set to contain pq. Thus our partition of $S(k + 1)$ in fact has at least $k - 1 + 2 + 1 = k + 2$ subsets, and not $k + 1$ subsets as we wanted. The claim is thus proved.

Returning to our original problem, since $p_{14} = 43 < \sqrt{2022}$, any partition of S must restrict to a partition of $S(14)$, which we showed must have at least 15 sets. Thus, we can do no better than 15. $\square$