24th Philippine Mathematical Olympiad

Qualifying Stage, 19 February 2022

PART I. Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

1. Let XZ be a diameter of circle ω. Let Y be a point on XZ such that $XY = 7$ and $YZ = 1$. Let W be a point on ω such that WY is perpendicular to XZ. What is the square of the length of the line segment WY?

(a) 7 (b) 8 (c) 10 (d) 25

2. How many five-digit numbers containing each of the digits 1, 2, 3, 4, 5 exactly once are divisible by 24?

(a) 8 (b) 10 (c) 12 (d) 20

3. A lattice point is a point (x, y) where x and y are both integers. Find the number of lattice points that lie on the closed line segment whose endpoints are $(2002, 2022)$ and $(2022, 2202)$.

(a) 20 (b) 21 (c) 22 (d) 23

4. Let $\omega \neq -1$ be a complex root of $x^3 + 1 = 0$. What is the value of $1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^4$?

(a) 3 (b) -4 (c) 5 (d) -6

5. How many ending zeroes does the decimal expansion of 2022! have?

(a) 404 (b) 484 (c) 500 (d) 503

6. Two tigers, Alice and Betty, run in the same direction around a circular track of circumference 400 meters. Alice runs at a speed of 10 m/s and Betty runs at 15 m/s. Betty gives Alice a 40 meter headstart before they both start running. After 15 minutes, how many times will they have passed each other?

(a) 9 (b) 10 (c) 11 (d) 12

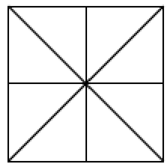
7. Suppose a, b, c are the roots of the polynomial $x^3 + 2x^2 + 2$. Let f be the unique monic polynomial whose roots are a^2, b^2, c^2. Find $f(1)$. (Note: A *monic* polynomial is a polynomial whose leading coefficient is 1.)

(a) -17 (b) -16 (c) -15 (d) -14

8. Let I be the center of the incircle of triangle ABC. Suppose that this incircle has radius 3, and that $AI = 5$. If the area of the triangle is 2022, what is the length of BC?

(a) 670 (b) 672 (c) 1340 (d) 1344

9. A square is divided into eight triangles as shown below. How many ways are there to shade exactly three of them so that no two shaded triangles share a common edge?



(a) 12 (b) 16 (c) 24 (d) 30

10. The numbers $2, b, c, d, 72$ are listed in increasing order so that $2, b, c$ form an arithmetic sequence, b, c, d form a geometric sequence, and $c, d, 72$ form a harmonic sequence (that is, a sequence whose reciprocals of its terms form an arithmetic sequence). What is the value of $b + c$?

(a) 7 (b) 13 (c) 19 (d) 25

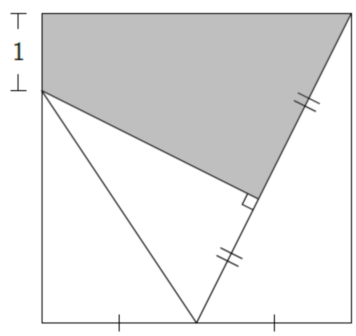
11. How many positive integers $n < 2022$ are there for which the sum of the odd positive divisors of n is 24?

(a) 7 (b) 8 (c) 14 (d) 15

12. Call a whole number *ordinary* if the product of its digits is less than or equal to the sum of its digits. How many numbers from the set $\{1, 2, \ldots, 999\}$ are ordinary?

(a) 151 (b) 162 (c) 230 (d) 241

13. What is the area of the shaded region of the square below?



(a) 7 (b) 11 (c) 15 (d) 19

14. Bryce plays a game in which he flips a fair coin repeatedly. In each flip, he obtains two tokens if the coin lands on heads, and loses one token if the coin lands on tails. At the start, Bryce has nine tokens. If after nine flips, he also ends up with nine tokens, what is the probability that Bryce always had at least nine tokens?

(a) 1/7 (b) 1/6 (c) 5/28 (d) 17/84

15. How many ways are there to arrange the first ten positive integers such that the multiples of 2 appear in increasing order, and the multiples of 3 appear in decreasing order?

(a) 720 (b) 2160 (c) 5040 (d) 6480

PART II. All answers are positive integers. Do not use commas if there are more than 3 digits, e.g., type 1234 instead of $1,234$. A fraction a/b is in lowest terms if a and b are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. What is the largest multiple of 7 less than 10,000 which can be expressed as the sum of squares of three consecutive numbers?

17. Suppose that the polynomial $P(x) = x^3 + 4x^2 + bx + c$ has a single root r and a double root s for some distinct real numbers r and s. Given that $P(-2s) = 324$, what is the sum of all possible values of $|c|$?

18. Let m and n be relatively prime positive integers. If m^3n^5 has 209 positive divisors, then how many positive divisors does m^5n^3 have?

19. Let x be a positive real number. What is the maximum value of $\dfrac{2022x^2\log(x+2022)}{(\log(x+2022))^3 + 2x^3}$?

20. Let a, b, c be real numbers such that

$$3ab + 2 = 6b, \quad 3bc + 2 = 5c, \quad 3ca + 2 = 4a.$$

Suppose the only possible values for the product abc are r/s and t/u, where r/s and t/u are both fractions in lowest terms. Find $r + s + t + u$.

21. You roll a fair 12-sided die repeatedly. The probability that all the primes show up at least once before seeing any of the other numbers can be expressed as a fraction p/q in lowest terms. What is $p + q$?

22. Let PMO be a triangle with $PM = 2$ and $\angle PMO = 120^\circ$. Let B be a point on PO such that PM is perpendicular to MB, and suppose that $PM = BO$. The product of the lengths of the sides of the triangle can be expressed in the form $a + b\sqrt[3]{c}$, where a, b, c are positive integers, and c is minimized. Find $a + b + c$.

23. Let ABC be a triangle such that the altitude from A, the median from B, and the internal angle bisector from C meet at a single point. If $BC = 10$ and $CA = 15$, find AB^2.

24. Find the sum of all positive integers n, $1 \le n \le 5000$, for which

$$n^2 + 2475n + 2454 + (-1)^n$$

is divisible by 2477. (Note that 2477 is a prime number.)

25. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. Consider the function

$$f(x, y) = \sqrt{M(M+1)}\Big(|x - m| + |y - m|\Big),$$

where $M = \max(\lfloor x \rfloor, \lfloor y \rfloor)$ and $m = \min(\lfloor x \rfloor, \lfloor y \rfloor)$. The set of all real numbers (x, y) such that $2 \le x, y \le 2022$ and $f(x, y) \le 2$ can be expressed as a finite union of disjoint regions in the plane. The sum of the areas of these regions can be expressed as a fraction a/b in lowest terms. What is the value of $a + b$?

Answers

Part I. (2 points each)

1. A	**6.** D	**11.** D
2. B	**7.** C	**12.** D
3. B	**8.** A	**13.** D
4. D	**9.** B	**14.** A
5. D	**10.** C	**15.** D

Part II. (5 points each)

16. $\boxed{8750}$

Let the number be expressed as $a^2 + (a+1)^2 + (a+2)^2$, where a is an integer. It may be checked that this expression is a multiple of 7 if and only if the remainder when a is divided by 7 is 1 or 4. In the former case, the largest possible value of a that places the value of the expression within bounds is 50, which gives the value $50^2 + 51^2 + 52^2 = 7805$. In the latter case, the largest such value of a is 53, which gives the value $53^2 + 54^2 + 55^2 = 8750$.

17. $\boxed{108}$

By Vieta's formula, we have $r + 2s = -4$ and writing $P(x) = (x-r)(x-s)^2$, we have $324 = P(-2s) = (-2s-r)(-3s)^2 = 36s^2$. Thus, $s^2 = 9$ and $s \in \{-3, 3\}$. We next observe that $|c| = |P(0)| = |r|s^2 = |-4-2s|s^2$. Hence, the sum of all possible values of $|c|$ is $9(|-4-6| + |-4+6| = 108$.

18. $\boxed{217}$

Let $d(N)$ denote the number of positive divisors of an integer N. Suppose that the prime factorizations of m and n are $\prod(p_i^{a_i})$ and $\prod(q_i^{b_i})$ respectively. Observe that 209 has four positive divisors: $1, 11, 19, 209$. If $m = 1$, then n^5 would have 209 divisors. Thus, $d(n^5) = 209$. However, it is known that $d(n^5) = \prod(5b_i + 1)$, but the latter implies that the remainder of $d(n^5)$ when divided by 5 is 1, a contradiction.

Likewise, if $n = 1$, then it implies that the remainder when $d(m^3) = 209$ is divided by 3 is 1, also a contradiction.

Thus, $m, n > 1$, so $d(m), d(n) > 1$. As m and n are relatively prime, then $d(m^3n^5) = d(m^3)d(n^5)$. As the only way to factor 209 as a product of 2 integers greater than 1 is as $11 \cdot 19$, then $d(m^3)$ and $d(n^5)$ are 11 and 19 in some order. As the remainder when $d(m^3)$ and $d(n^5)$ is divided by 3 and 5 respectively is 1, then $d(m^3) = 19 = (3 \cdot 6 + 1)$, and $d(n^5) = 11 = (5 \cdot 2 + 1)$. Thus, m and n can be expressed as p^6 and q^2 respectively. Therefore, $d(m^5n^3) = d(p^{30}q^6) = (30+1)(6+1) = 217$.

19. $\boxed{674}$

By the AM-GM Inequality,

$$\begin{aligned}
\frac{2022x^2\log(x+2022)}{\log^3(x+2022)+2x^3} &= \frac{2022x^2\log(x+2022)}{\log^3(x+2022)+x^3+x^3} \\
&\leq \frac{2022x^2\log(x+2022)}{3(x^6\log^3(x+2022))^{\frac{1}{3}}} \\
&= \frac{2022x^2\log(x+2022)}{3x^2\log(x+2022)} \\
&= 674
\end{aligned}$$

Equality holds when $x = \log(x + 2022)$, which has a positive solution.

20. $\boxed{18}$

The three given equations can be written as

$$3a + \frac{2}{b} = 12, \quad 3b + \frac{2}{c} = 10, \quad 3c + \frac{2}{a} = 8.$$

The product of all the three equations gives us

$$27abc + 6\left(3a + \frac{2}{b}\right) + 6\left(3b + \frac{2}{c}\right) + 6\left(3c + \frac{2}{a}\right) + \frac{8}{abc} = 120.$$

Plugging in the values and simplifying the equation gives us

$$27(abc)^2 - 30abc + 8 = 0.$$

This gives abc as either $4/9$ or $2/3$, so $r + s + t + u = 4 + 9 + 2 + 3 = 18$.

21. $\boxed{793}$

There are 5 primes which are at most 12 – namely 2,3,5,7 and 11. Notice that if a number has already been seen, we can effectively ignore all future occurrences of the number. Thus, the desired probability is the fraction of the permutations of $(1, 2, \ldots, 12)$ such that the primes all occur first. There are 5! ways to arrange the primes, and 7! ways to arrange the composites to satisfy the condition. Since there are 12! possible permutations, the desired probability is $\frac{5!7!}{12!} = \frac{1}{792}$, and the required sum is $1 + 792 = 793$.

22. $\boxed{28}$

Extend PM to a point C such that $PC \perp OC$. Since $\angle PMO = 120°$, $\angle CMO = 60°$ and $\angle COM = 30°$. Let $PB = x$ and $MC = a$. Then $CO = a\sqrt{3}$ and $OM = 2a$. Moreover, $\triangle PMB$ and $\triangle PCO$ are similar triangles. Thus, we have

$$\frac{2}{2+a} = \frac{x}{x+2}$$

so $x = 4/a$.

Furthermore, by the Cosine Law on side PO of $\triangle PMO$, we have

$$(x+2)^2 = 4 + 4a^2 + 2a$$

Plugging in $x = 4/a$ and expanding, we have

$$\frac{16}{a^2} + \frac{16}{a} + 4 = 4 + 4a^2 + 4a$$

and so $4 + 4a = a^4 + a^3$. Hence $a^3 = 4$ and $a = \sqrt[3]{4}$. Thus, $x = 4/\sqrt[2]{4}$.

It follows that the product of the lengths of the sides of the triangle is

$$(2a)(2)(x+2) = (2\sqrt[3]{4})(2)(2\sqrt[3]{2} + 2) = 16 + 8\sqrt[3]{4},$$

so $a + b + c = 16 + 8 + 4 = 28$.

23. $\boxed{205}$

Let D be the foot of the A-altitude, E the midpoint of AC, and F the foot of the C-internal angle bisector. Then by Ceva's Theorem, we have $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, and so $\frac{b}{a} \cdot \frac{c\cos B}{b\cos C} = 1$, where we are using the shorthand $BC = a, CA = b, AB = c$. By cosine law, we know that $\cos C = \frac{a^2+b^2-c^2}{2ab}$ and $\cos B = \frac{c^2+a^2-b^2}{2ca}$. Substituting this into the equation, we obtain

$$a(a^2 + b^2 - c^2) = b(c^2 + a^2 - b^2)$$

Solving for $c^2 = AB^2$ in this equation then gives the final answer, which is 205.

24. $\boxed{9912}$

The problem is equivalent to finding all values of n that satisfies the following congruence:

$$n^2 - 2n - 23 + (-1)^n \equiv 0 \pmod{2477}.$$

If n is odd, then $(-1)^n = -1$. Then, we have:

$$n^2 - 2n - 24 \equiv 0 \pmod{2477}$$

$$(n-6)(n+4) \equiv 0 \pmod{2477}.$$

This gives us solutions $n \equiv 6$ and $n \equiv -4$ (modulo 2477). Since 2477 is prime, these are the only roots modulo 2477. This yields the solutions 6, 2483, 2473 and 4950. We only accept the odd values, 2483 and 2473.

If n is even, then $(-1)^n = 1$. Then, we have

$$n2 - 2n - 22 \equiv 0 \pmod{2477}.$$

This gives $n \equiv 1 \pm \sqrt{23} \pmod{2477}$. Again, because 2477 is prime, these two are the only roots, modulo 2477. Since $2477 + 23 = 2500 = 50^2$, note that $\sqrt{23} \equiv 50 \pmod{2477}$, which then gives the solutions $n \equiv 51$ and $n \equiv -49$ (modulo 2477), yielding us the solutions 51, 2528, 2428 and 4905. We only accept the even values, 2528 and 2428.

Thus, the sum of all the solutions is $2483 + 2473 + 2528 + 2428 = 9912$.

25. $\boxed{2021}$

Fix $m \geq 2$. First note that $x, y \geq m$, and that $M \geq 2$, implying $\sqrt{M(M+1)} \geq 2$. Thus, $f(x,y) \leq 2$ necessarily implies $|x-m| + |y-m| \leq 1$, and so $|x-m|, |y-m| \leq 1$. This implies $x \in [m-1, m+1]$, and so $x \in [m, m+1]$. Similarly, $y \in [m, m+1]$. Now if $x = m+1$, then $1 \leq |x-m| + |y-m| \leq \frac{2}{\sqrt{M(M+1)}} < 1$, contradiction. Using the same argument for y, it follows that $x, y \in [m, m+1)$. Thus, $\lfloor x \rfloor = \lfloor y \rfloor = m$ always, and so $M = m$.

The inequality is then equivalent to $|x-m| + |y-m| \leq \frac{2}{\sqrt{m(m+1)}}$. Let $r \leq 1$ be the upper bound in the inequality. Keeping in mind the fact that $x, y \geq m$, this region is a right-triangle with vertices (m,m), $(m+r, m)$ and $(m, m+r)$, which then has area $\frac{r^2}{2} = \frac{2}{m(m+1)}$. This region is within the set for $2 \leq m \leq 2021$, so the desired sum is

$$\sum_{m=2}^{2021} \frac{2}{m(m+1)} = 2\sum_{m=2}^{2021} \left(\frac{1}{m} - \frac{1}{m+1}\right) = 2\left(\frac{1}{2} - \frac{1}{2022}\right) = \frac{1010}{1011}.$$

It then follows that $a + b = 2021$.