THE 24TH PHILIPPINE

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MATHEMATICAL OLYMPIAD

THE 24TH PHILIPPINE

QUALIFYING STAGE

February 19, 2022

NATIONAL STAGE

March 18-19, 2022

AWARDING CEREMONY

March 20, 2022

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MATHEMATICAL OLYMPIAD

About the PMO

First held in **1984**, the **Philippine Mathematical Olympiad (PMO)** was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are:

1) to stimulate the improvement of mathematics education in the country by awakening greater interest in and appreciation of mathematics among students and teachers, and gaining insights into the levels of mathematical learning;
2) to identify and motivate the mathematically gifted;
3) to identify potential participants to the International Mathematical Olympiad;
4) to provide a vehicle for the professional growth of teachers; and
5) to encourage the involvement of both public and private sectors in the concerted promotion and development of mathematics education.

The PMO is only the first part of the selection program implemented by the Mathematical Society of the Philippines towards the country's participation in the **International Mathematical Olympiad (IMO).** The thirty national finalists of the PMO will be invited to the **Mathematical Olympiad Summer Camp (MOSC),** a training program where participants will experience problem solving at a level that will help them grow in mathematical maturity, in preparation for the IMO. The selection tests and quizzes will then determine the six contestants who will form the country's National Team in Mathematics - the Philippine Team to the International Mathematical Olympiad.

The Philippine Mathematical Olympiad, the Mathematical Olympiad Summer Camp, and the country's participation in the International Mathematical Olympiad, are projects of both the **Mathematical Society of the Philippines** and the **Department of Science and Technology - Science Education Institute.**

The PMO this year is the twenty-fourth since 1984. Because of the ongoing COVID-19 pandemic, this is the second year in a row that the PMO was **held online. Two thousand nine hundred fifteen (2,915)** contestants from Grades 7 to 12 participated in the Qualifying Stage last February 19, 2022, from which our **thirty-two national finalists,** along with other **special awardees,** were chosen. From the national finalists, the Philippine Team to the **63rd International Mathematical Olympiad** were chosen. The IMO this year will be held from **July 6 to 16** in **Oslo, Norway.** This year, the **IMO resumes onsite competitions** after being conducted online the previous two years.

Message

By Dr. Josette T. Biyo, Director, DOST-SEI

At the 24th Philippine Mathematical Olympiad

Greetings of safety and abundance!

The Department of Science and Technology – Science Education Institute takes pride in its partnership with the Mathematical Society of the Philippines in developing the mathematical minds of the Filipino youth through the Philippine Mathematical Olympiad.

The PMO, which is now in its 24th year, has become one of the most sought-after academic competitions for high school students in the country. Being the oldest and most prestigious nationwide mathematics competition, the PMO has honed and produced hundreds of talented mathematicians who are now making waves in the field of mathematics.

Last year, despite the challenges brought about by the pandemic, the PMO pushed through an online competition which was well-received by the contestants. The students, and their unending enthusiasm in competition and solving math problems, are our inspiration for continuing the work that we do in the Institute. The DOST-SEI will remain committed to its mandate to empower the Filipino youth to become scholars for the nation and be the best versions of themselves with meaningful contributions in the fields of Science, Technology, Engineering and Math.

It is important that we pursue the conduct of these kinds of competition and provide opportunities for our students to reach their maximum potential despite the changes the world is facing today. We must continue to help our students dream, compete, and win because the world is for them to conquer. Trust that DOST-SEI will be there to help students achieve their dreams to become the best in the field of STEM.

May this competition bring joy and excitement to all contestants and promote camaraderie amongst our students. Only through our collective efforts, expertise and intelligence will we be able to create a better future anchored in science and a development that is for all.

JOSETTE T. BIYO

Director, Science Education Institute
Department of Science and Technology

Message

By Dr. Jose Ernie C. Lope, President, MSP

At the 24th Philippine Mathematical Olympiad

Congratulations to all the participants of the 24th edition of the Philippine Mathematical Olympiad, most especially to the national finalists and winners! Congratulations also to the PMO Organizing Team led by Dr. Richard Eden, all the Regional Coordinators, and the Test Development Committee headed by Dr. Christian Paul Chan Shio for the successful conduct of the Olympiad despite the challenges posed by the pandemic.

This year's set of finalists is particularly remarkable — more regions are now represented, and we have finalists coming from Luzon, Visayas, and Mindanao! Compared to the final rounds in recent memory, we also have many more females this year. We look forward to having more female finalists and more regional representatives in the years to come.

The Mathematical Society of the Philippines is committed to provide avenues for its members and for the larger Philippine mathematical community to pursue their research interests, disseminate their findings, and further enhance their mathematical skills. This commitment naturally includes the nurturing of young Filipinas and Filipinos, who we hope will pursue careers in mathematics and its allied fields.

What sets the PMO apart from the other math competitions is the quality of its questions — formulated by seasoned veterans and designed to identify potential members of the Philippine Team. The MSP is most certainly pleased with our Team's performance in the Asia Pacific and International Mathematical Olympiads in recent years. With the continued support of the DOST-SEI, we at the MSP are confident that our recent accomplishments will be sustained, if not surpassed.

Padayon!

JOSE ERNIE C. LOPE

President
Mathematical Society of the Philippines

The PMO Team

National Level Administrators

Director	Dr. Richard B. Eden Department of Mathematics, Ateneo de Manila University
Assistant Director	Dr. Ezra S. Aguilar Department of Mathematics and Physics, University of Santo Tomas
Secretary	Dr. Daryl Allen B. Saddi Institute of Mathematics, University of the Philippines Diliman
Treasurer	Mr. Calvin S. Sia Department of Mathematics, Ateneo de Manila University
Test Development Committee Head	Dr. Christian Paul O. Chan Shio Department of Mathematics, Ateneo de Manila University

Regional Coordinators

Region 1 & CAR	Mr. Anthony M. Pasion Department of Mathematics and Computer Science, University of the Philippines – Baguio
Region 2	Mr. Crizaldy P. Binarao Department of Natural Sciences and Mathematics, Cagayan State University
Region 3	Dr. Nancy L. Mati Mathematics Department, Tarlac State University
Region 4A	Ms. Sharon P. Lubag Mathematics and Statistics Department, De La Salle University – Dasmariñas

Regional Coordinators

Region 4B	Ms. Emmalyn T. Venturillo College of Arts and Sciences, Western Philippines University
Region 5	Ms. Glenda G. Quinto Department of Mathematics, Ateneo de Naga University
Region 6	Dr. Alexander J. Balsomo Department of Mathematics, West Visayas State University
Region 7	Dr. Cherrylyn P. Alota Department of Mathematics and Statistics, University of the Philippines — Cebu
Region 8	Mr. Oreste M. Ortega, Jr. Mathematics Department, Leyte Normal University
Region 9	Mr. Dante V. Partosa Mathematics Department, Ateneo de Zamboanga University
Region 10	Mr. Paolo B. Araune Mathematics Department, Xavier University – Ateneo de Cagayan
Region 11	Mr. Joseph T. Belida Mathematics Department, Ateneo de Davao University
Region 12, 13, & BARMM	Dr. Miraluna L. Herrera Department of Mathematics, Caraga State University
NCR	Dr. Kristine Joy E. Carpio Mathematics and Statistics Department, De La Salle University
	Dr. Jude C. Buot Department of Mathematics, Ateneo de Manila University

Special Awards

Top Contestant Per Region from the Qualifying Stage

NCR

Raphael Dylan T. Dalida
Philippine Science High School - Main Campus

Jerome Austin N. Te
Jubilee Christian Academy

CAR

Han Na Park
Philippine Science High School - Cordillera Administrative Region Campus

Karl Uriel Dela Cruz
Philippine Science High School - Cordillera Administrative Region Campus

Region 1

Phylline Cristel O. Calubayan
BHC Educational Institution, Inc.

Region 2

Grace B. Finotton
Maddela Comprehensive High School

Region 3

Justin Teng Soon T. Khoo
Regional Science High School III

Region 4A

Jan Neal Isaac D. Villamin
Santa Rosa Science and Technology High School

Region 5

Carl Dexter N. Donor
Sorsogon National High School

John Angelo O. Oringo
Philippine Science High School - Bicol Region Campus

Region 6

Jonathan D. Anacan
Philippine Science High School - Western Visayas Campus

Special Awards

Top Contestant Per Region from the Qualifying Stage

Region 7
Lance Christopher M. Jimenez
San Roque College de Cebu - Liloan Campus

Region 8
Evan Justin E. Panergo
Calbayog City National High School

Region 9
Douglas Frederick S. Guangco
Zamboanga Chong Hua High School

Region 10
Mohammad Nur G. Casib
Philippine Science High School - Central Mindanao Campus

Region 11
Leonardo Florenz S. Eugenio
Philippine Science High School - Southern Mindanao Campus

Region 12
Felinwright Niñokyle A. Mesias
Philippine Science High School - SOCCSKSARGEN Region Campus

Region 13
Matt Raymond Ayento
Philippine Science High School - Caraga Region Campus

BARMM
Mohammad Miadh M. Angkad
Albert Einstein School, Inc.

Special Awards

Top Contestant Per Area from the Qualifying Stage

Luzon

1st Justin Teng Soon T. Khoo
Regional Science High School III

2nd Jan Neal Isaac D. Villamin
Santa Rosa Science and Technology High School

3rd Rainier M. Guinto
Cavite Science Integrated School

3rd Justin M. Sarmiento
Marcelo H. Del Pilar National High School

Visayas

1st Jonathan D. Anacan
Philippine Science High School - Western Visayas Campus

2nd Christopher James C. Yap
St. John's Institute

3rd Evan Justin E. Panergo
Calbayog City National High School

3rd Christian Jacob C. Yap
St. John's Institute

Mindanao

1st Mohammad Nur G. Casib
Philippine Science High School - Central Mindanao Campus

2nd Mohammad Miadh M. Angkad
Albert Einstein School, Inc.

2nd Matt Raymond Ayento
Philippine Science High School - Caraga Region Campus

NCR

1st Raphael Dylan T. Dalida
Philippine Science High School - Main Campus

1st Jerome Austin N. Te
Jubilee Christian Academy

3rd Luke Sebastian C. Sy
Grace Christian College

3rd Shawn Darren S. Chua
MGC New Life Christian Academy

3rd Alvann Walter W. Paredes Dy
Saint Jude Catholic School

Special Awards

Top Female Contestant

Kristen Steffi S. Teh
Ateneo de Manila
Senior High School

Top Junior Contestants

Mohammad Nur G. Casib
Philippine Science High School -
Central Mindanao Campus

Jerome Austin N. Te
Jubilee Christian Academy

National Finalists

Karl Francois Dwayne N. Agudong
Quezon City Science High School

Deanne Gabrielle D. Algenio
Ateneo de Manila Senior High School

Jonathan D. Anacan
Philippine Science High School - Western Visayas Campus

Mohammad Miadh M. Angkad
Albert Einstein School, Inc.

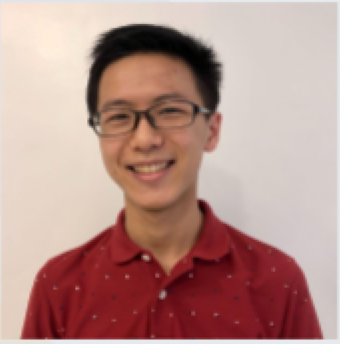
Matt Raymond Ayento
Philippine Science High School - Caraga Region Campus

Dominic Lawrence R. Bermudez
Philippine Science High School - Main Campus

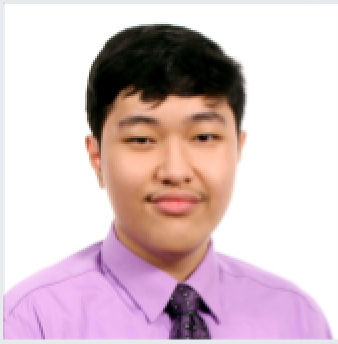
Jose Maria S. Bernardo II
Ateneo de Manila Senior High School

Mohammad Nur G. Casib
Philippine Science High School - Central Mindanao Campus

Enzo Rafael S. Chan
Ateneo de Manila Senior High School



Shawn Darren S. Chua
MGC New Life Christian Academy



Raphael Dylan T. Dalida
Philippine Science High School - Main Campus

National Finalists

Alexandra Brianne B. Gochian
Saint Jude Catholic School

Benjamin L. Jacob
Philippine Science High School - Main Campus

Justin Teng Soon T. Khoo
Regional Science High School III

Hashanti Nikisha T. Liao
St. Stephen's High School

Enrico Rolando G. Martinez
Philippine Science High School - Main Campus

Martin Johan M. Ocho
De La Salle University Integrated School

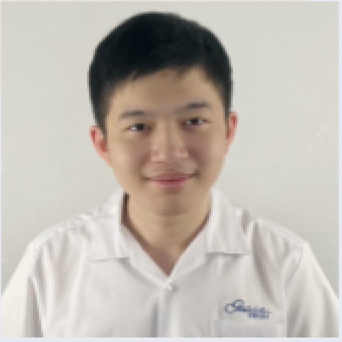
Citrei Kim K. Padayao
Quezon City Science High School

Wesley Gavin G. Palomar
Philippine Science High School - Main Campus

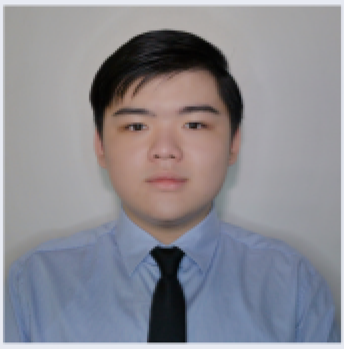
Erich A. Paredes
International School Manila

Alvann Walter W. Paredes Dy
Saint Jude Catholic School

National Finalists



Luke Sebastian C. Sy
Grace Christian College



Cassidy Kyler L. Tan
Ateneo de Manila Senior High School

Frederick Ivan T. Tan
Philippine Science High School - Main Campus

Patricia Angelica P. Tan
De La Salle University Integrated School

Rickson Caleb Y. Tan
MGC New Life Christian Academy

Sean Matthew G. Tan
Jubilee Christian Academy

Keean Nathaniel T. Tang
Ateneo de Manila Senior High School

Jerome Austin N. Te
Jubilee Christian Academy

Kristen Steffi S. Teh
Ateneo de Manila Senior High School

Jan Neal Isaac D. Villamin
Santa Rosa Science and Technology High School

Filbert Ephraim S. Wu
Victory Christian International School

24th Philippine Mathematical Olympiad

Qualifying Stage, 19 February 2022

PART I. Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

1. Let XZ be a diameter of circle ω. Let Y be a point on XZ such that $XY = 7$ and $YZ = 1$. Let W be a point on ω such that WY is perpendicular to XZ. What is the square of the length of the line segment WY?

(a) 7 (b) 8 (c) 10 (d) 25

2. How many five-digit numbers containing each of the digits 1, 2, 3, 4, 5 exactly once are divisible by 24?

(a) 8 (b) 10 (c) 12 (d) 20

3. A lattice point is a point (x, y) where x and y are both integers. Find the number of lattice points that lie on the closed line segment whose endpoints are $(2002, 2022)$ and $(2022, 2202)$.

(a) 20 (b) 21 (c) 22 (d) 23

4. Let $\omega \neq -1$ be a complex root of $x^3 + 1 = 0$. What is the value of $1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^4$?

(a) 3 (b) -4 (c) 5 (d) -6

5. How many ending zeroes does the decimal expansion of 2022! have?

(a) 404 (b) 484 (c) 500 (d) 503

6. Two tigers, Alice and Betty, run in the same direction around a circular track of circumference 400 meters. Alice runs at a speed of 10 m/s and Betty runs at 15 m/s. Betty gives Alice a 40 meter headstart before they both start running. After 15 minutes, how many times will they have passed each other?

(a) 9 (b) 10 (c) 11 (d) 12

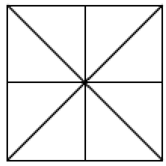
7. Suppose a, b, c are the roots of the polynomial $x^3 + 2x^2 + 2$. Let f be the unique monic polynomial whose roots are a^2, b^2, c^2. Find $f(1)$. (Note: A *monic* polynomial is a polynomial whose leading coefficient is 1.)

(a) -17 (b) -16 (c) -15 (d) -14

8. Let I be the center of the incircle of triangle ABC. Suppose that this incircle has radius 3, and that $AI = 5$. If the area of the triangle is 2022, what is the length of BC?

(a) 670 (b) 672 (c) 1340 (d) 1344

9. A square is divided into eight triangles as shown below. How many ways are there to shade exactly three of them so that no two shaded triangles share a common edge?



(a) 12 (b) 16 (c) 24 (d) 30

10. The numbers $2, b, c, d, 72$ are listed in increasing order so that $2, b, c$ form an arithmetic sequence, b, c, d form a geometric sequence, and $c, d, 72$ form a harmonic sequence (that is, a sequence whose reciprocals of its terms form an arithmetic sequence). What is the value of $b + c$?

(a) 7 (b) 13 (c) 19 (d) 25

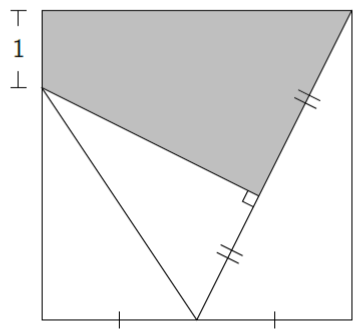
11. How many positive integers $n < 2022$ are there for which the sum of the odd positive divisors of n is 24?

(a) 7 (b) 8 (c) 14 (d) 15

12. Call a whole number *ordinary* if the product of its digits is less than or equal to the sum of its digits. How many numbers from the set $\{1, 2, \ldots, 999\}$ are ordinary?

(a) 151 (b) 162 (c) 230 (d) 241

13. What is the area of the shaded region of the square below?



(a) 7 (b) 11 (c) 15 (d) 19

14. Bryce plays a game in which he flips a fair coin repeatedly. In each flip, he obtains two tokens if the coin lands on heads, and loses one token if the coin lands on tails. At the start, Bryce has nine tokens. If after nine flips, he also ends up with nine tokens, what is the probability that Bryce always had at least nine tokens?

(a) 1/7 (b) 1/6 (c) 5/28 (d) 17/84

15. How many ways are there to arrange the first ten positive integers such that the multiples of 2 appear in increasing order, and the multiples of 3 appear in decreasing order?

(a) 720 (b) 2160 (c) 5040 (d) 6480

PART II. All answers are positive integers. Do not use commas if there are more than 3 digits, e.g., type 1234 instead of $1,234$. A fraction a/b is in lowest terms if a and b are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. What is the largest multiple of 7 less than 10,000 which can be expressed as the sum of squares of three consecutive numbers?

17. Suppose that the polynomial $P(x) = x^3 + 4x^2 + bx + c$ has a single root r and a double root s for some distinct real numbers r and s. Given that $P(-2s) = 324$, what is the sum of all possible values of $|c|$?

18. Let m and n be relatively prime positive integers. If m^3n^5 has 209 positive divisors, then how many positive divisors does m^5n^3 have?

19. Let x be a positive real number. What is the maximum value of $\dfrac{2022x^2\log(x+2022)}{(\log(x+2022))^3 + 2x^3}$?

20. Let a, b, c be real numbers such that

$$3ab + 2 = 6b, \quad 3bc + 2 = 5c, \quad 3ca + 2 = 4a.$$

Suppose the only possible values for the product abc are r/s and t/u, where r/s and t/u are both fractions in lowest terms. Find $r + s + t + u$.

21. You roll a fair 12-sided die repeatedly. The probability that all the primes show up at least once before seeing any of the other numbers can be expressed as a fraction p/q in lowest terms. What is $p + q$?

22. Let PMO be a triangle with $PM = 2$ and $\angle PMO = 120^\circ$. Let B be a point on PO such that PM is perpendicular to MB, and suppose that $PM = BO$. The product of the lengths of the sides of the triangle can be expressed in the form $a + b\sqrt[3]{c}$, where a, b, c are positive integers, and c is minimized. Find $a + b + c$.

23. Let ABC be a triangle such that the altitude from A, the median from B, and the internal angle bisector from C meet at a single point. If $BC = 10$ and $CA = 15$, find AB^2.

24. Find the sum of all positive integers n, $1 \le n \le 5000$, for which

$$n^2 + 2475n + 2454 + (-1)^n$$

is divisible by 2477. (Note that 2477 is a prime number.)

25. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. Consider the function

$$f(x, y) = \sqrt{M(M+1)}\Big(|x - m| + |y - m|\Big),$$

where $M = \max(\lfloor x \rfloor, \lfloor y \rfloor)$ and $m = \min(\lfloor x \rfloor, \lfloor y \rfloor)$. The set of all real numbers (x, y) such that $2 \le x, y \le 2022$ and $f(x, y) \le 2$ can be expressed as a finite union of disjoint regions in the plane. The sum of the areas of these regions can be expressed as a fraction a/b in lowest terms. What is the value of $a + b$?

Answers

Part I. (2 points each)

1. A	**6**. D	**11**. D
2. B	**7**. C	**12**. D
3. B	**8**. A	**13**. D
4. D	**9**. B	**14**. A
5. D	**10**. C	**15**. D

Part II. (5 points each)

16. $\boxed{8750}$

Let the number be expressed as $a^2 + (a+1)^2 + (a+2)^2$, where a is an integer. It may be checked that this expression is a multiple of 7 if and only if the remainder when a is divided by 7 is 1 or 4. In the former case, the largest possible value of a that places the value of the expression within bounds is 50, which gives the value $50^2 + 51^2 + 52^2 = 7805$. In the latter case, the largest such value of a is 53, which gives the value $53^2 + 54^2 + 55^2 = 8750$.

17. $\boxed{108}$

By Vieta's formula, we have $r + 2s = -4$ and writing $P(x) = (x-r)(x-s)^2$, we have $324 = P(-2s) = (-2s-r)(-3s)^2 = 36s^2$. Thus, $s^2 = 9$ and $s \in \{-3, 3\}$. We next observe that $|c| = |P(0)| = |r|s^2 = |-4-2s|s^2$. Hence, the sum of all possible values of $|c|$ is $9(|-4-6| + |-4+6| = 108$.

18. $\boxed{217}$

Let $d(N)$ denote the number of positive divisors of an integer N. Suppose that the prime factorizations of m and n are $\prod(p_i^{a_i})$ and $\prod(q_i^{b_i})$ respectively. Observe that 209 has four positive divisors: $1, 11, 19, 209$. If $m = 1$, then n^5 would have 209 divisors. Thus, $d(n^5) = 209$. However, it is known that $d(n^5) = \prod(5b_i + 1)$, but the latter implies that the remainder of $d(n^5)$ when divided by 5 is 1, a contradiction.

Likewise, if $n = 1$, then it implies that the remainder when $d(m^3) = 209$ is divided by 3 is 1, also a contradiction.

Thus, $m, n > 1$, so $d(m), d(n) > 1$. As m and n are relatively prime, then $d(m^3n^5) = d(m^3)d(n^5)$. As the only way to factor 209 as a product of 2 integers greater than 1 is as $11 \cdot 19$, then $d(m^3)$ and $d(n^5)$ are 11 and 19 in some order. As the remainder when $d(m^3)$ and $d(n^5)$ is divided by 3 and 5 respectively is 1, then $d(m^3) = 19 = (3 \cdot 6 + 1)$, and $d(n^5) = 11 = (5 \cdot 2 + 1)$. Thus, m and n can be expressed as p^6 and q^2 respectively. Therefore, $d(m^5n^3) = d(p^{30}q^6) = (30+1)(6+1) = 217$.

19. $\boxed{674}$

By the AM-GM Inequality,

$$\begin{aligned}
\frac{2022x^2 \log(x+2022)}{\log^3(x+2022) + 2x^3} &= \frac{2022x^2 \log(x+2022)}{\log^3(x+2022) + x^3 + x^3} \\
&\leq \frac{2022x^2 \log(x+2022)}{3(x^6 \log^3(x+2022))^{\frac{1}{3}}} \\
&= \frac{2022x^2 \log(x+2022)}{3x^2 \log(x+2022)} \\
&= 674
\end{aligned}$$

Equality holds when $x = \log(x + 2022)$, which has a positive solution.

20. $\boxed{18}$

The three given equations can be written as

$$3a + \frac{2}{b} = 12, \quad 3b + \frac{2}{c} = 10, \quad 3c + \frac{2}{a} = 8.$$

The product of all the three equations gives us

$$27abc + 6\left(3a + \frac{2}{b}\right) + 6\left(3b + \frac{2}{c}\right) + 6\left(3c + \frac{2}{a}\right) + \frac{8}{abc} = 120.$$

Plugging in the values and simplifying the equation gives us

$$27(abc)^2 - 30abc + 8 = 0.$$

This gives abc as either $4/9$ or $2/3$, so $r + s + t + u = 4 + 9 + 2 + 3 = 18$.

21. $\boxed{793}$

There are 5 primes which are at most 12 – namely 2,3,5,7 and 11. Notice that if a number has already been seen, we can effectively ignore all future occurrences of the number. Thus, the desired probability is the fraction of the permutations of $(1, 2, \dots, 12)$ such that the primes all occur first. There are 5! ways to arrange the primes, and 7! ways to arrange the composites to satisfy the condition. Since there are 12! possible permutations, the desired probability is $\dfrac{5!7!}{12!} = \dfrac{1}{792}$, and the required sum is $1 + 792 = 793$.

22. $\boxed{28}$

Extend PM to a point C such that $PC \perp OC$. Since $\angle PMO = 120°$, $\angle CMO = 60°$ and $\angle COM = 30°$. Let $PB = x$ and $MC = a$. Then $CO = a\sqrt{3}$ and $OM = 2a$. Moreover, $\triangle PMB$ and $\triangle PCO$ are similar triangles. Thus, we have

$$\frac{2}{2+a} = \frac{x}{x+2}$$

so $x = 4/a$.

Furthermore, by the Cosine Law on side PO of $\triangle PMO$, we have

$$(x+2)^2 = 4 + 4a^2 + 2a$$

Plugging in $x = 4/a$ and expanding, we have

$$\frac{16}{a^2} + \frac{16}{a} + 4 = 4 + 4a^2 + 4a$$

and so $4 + 4a = a^4 + a^3$. Hence $a^3 = 4$ and $a = \sqrt[3]{4}$. Thus, $x = 4/\sqrt[2]{4}$.

It follows that the product of the lengths of the sides of the triangle is

$$(2a)(2)(x+2) = (2\sqrt[3]{4})(2)(2\sqrt[3]{2} + 2) = 16 + 8\sqrt[3]{4},$$

so $a + b + c = 16 + 8 + 4 = 28$.

23. $\boxed{205}$

Let D be the foot of the A-altitude, E the midpoint of AC, and F the foot of the C-internal angle bisector. Then by Ceva's Theorem, we have $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, and so $\frac{b}{a} \cdot \frac{c\cos B}{b\cos C} = 1$, where we are using the shorthand $BC = a, CA = b, AB = c$. By cosine law, we know that $\cos C = \frac{a^2+b^2-c^2}{2ab}$ and $\cos B = \frac{c^2+a^2-b^2}{2ca}$. Substituting this into the equation, we obtain

$$a(a^2 + b^2 - c^2) = b(c^2 + a^2 - b^2)$$

Solving for $c^2 = AB^2$ in this equation then gives the final answer, which is 205.

24. $\boxed{9912}$

The problem is equivalent to finding all values of n that satisfies the following congruence:

$$n^2 - 2n - 23 + (-1)^n \equiv 0 \pmod{2477}.$$

If n is odd, then $(-1)^n = -1$. Then, we have:

$$n^2 - 2n - 24 \equiv 0 \pmod{2477}$$

$$(n-6)(n+4) \equiv 0 \pmod{2477}.$$

This gives us solutions $n \equiv 6$ and $n \equiv -4$ (modulo 2477). Since 2477 is prime, these are the only roots modulo 2477. This yields the solutions 6, 2483, 2473 and 4950. We only accept the odd values, 2483 and 2473.

If n is even, then $(-1)^n = 1$. Then, we have

$$n2 - 2n - 22 \equiv 0 \pmod{2477}.$$

This gives $n \equiv 1 \pm \sqrt{23} \pmod{2477}$. Again, because 2477 is prime, these two are the only roots, modulo 2477. Since $2477 + 23 = 2500 = 50^2$, note that $\sqrt{23} \equiv 50 \pmod{2477}$, which then gives the solutions $n \equiv 51$ and $n \equiv -49$ (modulo 2477), yielding us the solutions 51, 2528, 2428 and 4905. We only accept the even values, 2528 and 2428.

Thus, the sum of all the solutions is $2483 + 2473 + 2528 + 2428 = 9912$.

25. $\boxed{2021}$

Fix $m \geq 2$. First note that $x, y \geq m$, and that $M \geq 2$, implying $\sqrt{M(M+1)} \geq 2$. Thus, $f(x,y) \leq 2$ necessarily implies $|x-m| + |y-m| \leq 1$, and so $|x-m|, |y-m| \leq 1$. This implies $x \in [m-1, m+1]$, and so $x \in [m, m+1]$. Similarly, $y \in [m, m+1]$. Now if $x = m+1$, then $1 \leq |x-m| + |y-m| \leq \frac{2}{\sqrt{M(M+1)}} < 1$, contradiction. Using the same argument for y, it follows that $x, y \in [m, m+1)$. Thus, $\lfloor x \rfloor = \lfloor y \rfloor = m$ always, and so $M = m$.

The inequality is then equivalent to $|x-m| + |y-m| \leq \frac{2}{\sqrt{m(m+1)}}$. Let $r \leq 1$ be the upper bound in the inequality. Keeping in mind the fact that $x, y \geq m$, this region is a right-triangle with vertices (m,m), $(m+r, m)$ and $(m, m+r)$, which then has area $\frac{r^2}{2} = \frac{2}{m(m+1)}$. This region is within the set for $2 \leq m \leq 2021$, so the desired sum is

$$\sum_{m=2}^{2021} \frac{2}{m(m+1)} = 2\sum_{m=2}^{2021}\left(\frac{1}{m} - \frac{1}{m+1}\right) = 2\left(\frac{1}{2} - \frac{1}{2022}\right) = \frac{1010}{1011}.$$

It then follows that $a + b = 2021$.

24^{th} Philippine Mathematical Olympiad

National Stage (Day 1)

18 March 2022

Time: 4.5 hours *Each item is worth 7 points.*

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers a, b, c, and d.

2. The PMO Magician has a special party game. There are n chairs, labelled 1 to n. There are n sheets of paper, labelled 1 to n.

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks n party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if $1 < m \leq n$, where m is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly m claps.

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer k, denote by S_k the set of all visible lattice points (x, y) such that $x^2 + y^2 = k^2$. Let D denote the set of all positive divisors of $2021 \cdot 2025$. Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point (x, y) on the plane where both x and y are integers, and $|A|$ denotes the number of elements of the set A.

4. Let $\triangle ABC$ have incenter I and centroid G. Suppose that P_A is the foot of the perpendicular from C to the exterior angle bisector of B, and Q_A is the foot of the perpendicular from B to the exterior angle bisector of C. Define P_B, P_C, Q_B, and Q_C similarly. Show that P_A, P_B, P_C, Q_A, Q_B, and Q_C lie on a circle whose center is on line IG.

24^{th} Philippine Mathematical Olympiad

National Stage (Day 2)

19 March 2022

Time: 4.5 hours *Each item is worth 7 points.*

5. Find all positive integers n for which there exists a set of exactly n distinct positive integers, none of which exceed n^2, whose reciprocals add up to 1.

6. In $\triangle ABC$, let D be the point on side BC such that $AB+BD = DC+CA$. The line AD intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. Prove that one of the common tangents of the circumcircles of $\triangle BDX$ and $\triangle CDX$ is parallel to BC.

7. Let a, b, and c be positive real numbers such that $ab + bc + ca = 3$. Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

8. The set $S = \{1, 2, \ldots, 2022\}$ is to be partitioned into n disjoint subsets $S_1, S_2, \ldots, S_n$ such that for each $i \in \{1, 2, \ldots, n\}$, exactly one of the following statements is true:

(a) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) > 1$.

(b) For all $x, y \in S_i$ with $x \neq y$, $\gcd(x, y) = 1$.

Find the smallest value of n for which this is possible.

24^{th} Philippine Mathematical Olympiad

National Stage (Solutions)

18-19 March 2022

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1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers a, b, c, and d.

Solution. We claim that the only solutions are $f(x) = 0$ and $f(x) = x$. It is easy to see that both satisfy the functional inequality. We now show that no other functions satisfy the given condition.

Letting $a = b = c = d$ yields $2f(0)^2 \leq 0$ which implies $f(0) = 0$. Now, suppose that $f \not\equiv 0$. Then we can find $p \in \mathbb{R}$ such that $f(p) \neq 0$. For $a \in \mathbb{R}$, let $b = a$, $c = 0$, and $d = a - p$. We then have

$$f(p)f(a) \leq af(p).$$

Case 1: Suppose $f(p) > 0$. Then $f(a) \leq a$ for all $a \in \mathbb{R}$. In particular $f(-1) < 0$. In the original functional equation, letting $b = a$, $c = 0$, and $d = a + 1$ yields $f(-1)f(a) \leq af(-1)$ for all $a \in \mathbb{R}$. Hence, $f(a) \geq a$ for all $a \in \mathbb{R}$. Thus for this case, $f(x) = x$ for all $x \in \mathbb{R}$.

Case 2: Suppose $f(p) < 0$. Then $f(a) \geq a$ for all $a \in \mathbb{R}$. In particular $f(1) > 0$. In the original functional equation, letting $b = a$, $c = 0$, and $d = a - 1$ yields $f(1)f(a) \leq af(1)$ for all $a \in \mathbb{R}$. Hence, $f(a) \leq a$ for all $a \in \mathbb{R}$. Thus for this case, $f(x) = x$ for all $x \in \mathbb{R}$.

□

2. The PMO Magician has a special party game. There are n chairs, labelled 1 to n. There are n sheets of paper, labelled 1 to n.

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks n party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if $1 < m \leq n$, where m is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly m claps.

Solution. Decompose the permutation into cycles of lengths $c_1, c_2, c_3, \ldots, c_k$. Note that $c_1 + c_2 + \cdots + c_k = n$. A guest in a cycle of length c returns to their original seat after c claps. Thus, all guests return to their original seats after $\operatorname{lcm}\{c_i\}$ claps.

Let $m = pq$, where p and q are coprime and both greater than 1; if $m > 1$ is not a prime power, then it has at least two different prime factors, so this is always possible. Suppose we have at least one cycle of lengths p and q, and *all* cycles are of lengths p or q. Then, all guests will return to their original seats after $\text{lcm}\{p, q\} = m$ claps.

If $m = n$, we can just connect all n party guests in one big cycle of size $m = n$. Otherwise, $m < n$, and note that the problem is equivalent to finding a nonnegative integer solution to $p\overline{x} + q\overline{y} = n - p - q$. Since $m < n$, a solution always exists by the Chicken McNugget Theorem, completing the proof. □

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer k, denote by S_k the set of all visible lattice points (x, y) such that $x^2 + y^2 = k^2$. Let D denote the set of all positive divisors of $2021 \cdot 2025$. Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point (x, y) on the plane where both x and y are integers, and $|A|$ denotes the number of elements of the set A.

Solution. We claim that the required sum is 20.

Let T_k denote the set of all lattice points in the circle $x^2 + y^2 = k^2$. We claim that $\sum_{d|k} |S_d| = |T_k|$. Indeed, given a point (x, y) in T_k, let $g = \gcd(x, y)$. Then $x/g, y/g$ are necessarily coprime, and hence $(x/g, y/g)$ visible, and $(x/g)^2 + (y/g)^2 = (k/g)^2$. This implies $(x/g, y/g) \in \bigcup_{d|k} S_d$. Next, note that the S_d's are necessarily disjoint. Now if (x', y') is a visible lattice point in S_d where $d|k$, then we can write $k = gd$ so that $(x, y) = (gx', gy')$ is a lattice point in T_k. This establishes a bijection between $\bigcup_{d|k} S_d$ and T_k, and since the S_d's are disjoint, the claim follows.

From the claim, it suffices to find the number of lattice points in the circle $x^2 + y^2 = (2021 \cdot 2025)^2$. This is equivalent to

$$x^2 + y^2 = 3^8 \cdot 5^4 \cdot 43^2 \cdot 47^2.$$

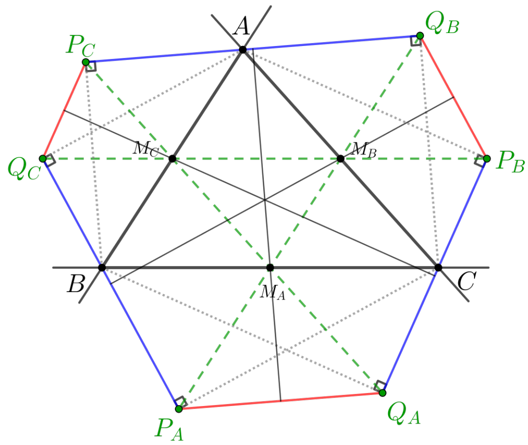
Now it is well-known that if $x^2 + y^2 \equiv 0 \pmod{p}$ where $p \equiv 3 \pmod{4}$ is a prime, then $x \equiv y \equiv 0 \pmod{p}$. Thus, we must also have $x, y \equiv 0 \pmod{3^4 \cdot 43 \cdot 47}$. It then follows that the number of lattice points is the same as the number of lattice points in $x^2 + y^2 = 25^2$.

If $x = 0$ or $y = 0$, there are 4 solutions. Otherwise, assume WLOG that they are both positive. Now it is well-known that all solutions to $x^2 + y^2 = z^2$ are in the form $x = g(m^2 - n^2)$, $y = 2gmn$, and $z = g(m^2 + n^2)$, where $m > n$ are coprime positive integers, and g is a positive integer. Thus, we want $g(m^2 + n^2) = 25$. Note that $g|25$, so $g = 1, 5, 25$.

If $g = 25$, then $m^2 + n^2 = 1$, so $n = 0$, contradiction. If $g = 5$, then $m^2 + n^2 = 5$, which yields $m = 2$ and $n = 1$ and thus $g(m^2 - n^2) = 15$ and $2gmn = 20$, so $(x, y) = (15, 20), (20, 15)$. If $g = 1$, then we get $m^2 + n^2 = 25$, from which we obtain $m = 4$ and $n = 3$. It then follows that $(x, y) = (24, 7), (7, 24)$, and so we have 2 solutions when x, y are both positive. This implies that there are $4 \cdot 4 = 16$ solutions when x, y are nonzero, and so there are $4 + 16 = 20$ solutions in total. □

4. Let $\triangle ABC$ have incenter I and centroid G. Suppose that P_A is the foot of the perpendicular from C to the exterior angle bisector of B, and Q_A is the foot of the perpendicular from B to the exterior angle bisector of C. Define P_B, P_C, Q_B, and Q_C similarly. Show that P_A, P_B, P_C, Q_A, Q_B, and Q_C lie on a circle whose center is on line IG.

Solution. Refer to the figure shown below:



Let $M_A, M_B,$ and M_C be the midpoints of $BC, CA,$ and AB respectively.

First, it may be shown that P_B and Q_C lie on M_BM_C.

Note that $\angle AM_CQ_C = 2\angle ABQ_C = 180° - \angle ABC = \angle BM_CM_B$. Thus, Q_C lies on M_BM_C. Likewise, $AM_BP_B = 2\angle ACP_B = 180° - \angle ACB = \angle CM_BM_C$. Thus, P_B also lies on M_BM_C.

Similarly, P_C and Q_A lie on M_CM_A, and P_A and Q_B lie on M_AM_B.

Now, as M_A is the center of the circle passing through $B, C, Q_A,$ and P_A, then $M_AP_A = M_AQ_A$, so the angle bisector of $\angle M_CM_AM_B$ coincides with the perpendicular bisector of P_AQ_A.

Observe that $M_CQ_A = M_CM_A + M_AQ_A = \frac{CA}{2} + \frac{BC}{2} = M_BP_B + M_CM_B = M_CP_B$. Thus, the perpendicular bisector of Q_AP_B coincides with the angle bisector of $\angle M_BM_CM_A$.

Using similar observations, it may then be concluded that the perpendicular bisectors of P_AQ_A, Q_AP_B, P_BQ_B, Q_BP_C, P_CQ_C, and Q_CP_A all concur at the incenter of $\triangle M_AM_BM_C$. Thus, the latter must also be the center of the circle containing all six points. As the medial triangle is the image of a homothety on $\triangle ABC$ with center G having a scale factor of -0.5, then the incenter of $\triangle M_AM_BM_C$ must lie on IG. □

5. Find all positive integers n for which there exists a set of exactly n distinct positive integers, none of which exceed n^2, whose reciprocals add up to 1.

Solution. The answer is all $n \neq 2$. For $n = 1$, the set $\{1\}$ works. For $n = 2$, no set exists, simply because the sum of reciprocals of two distinct integers cannot be equal to 1. For $n = 3$, take $\{2, 3, 6\}$.

For $n > 3$, the identity

$$\frac{1}{k} = \frac{1}{k+r} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(k+r-1)(k+r)}$$

allows us to extend a sum of t terms to one of exactly $t+r$ terms. Taking $k=3$ and $r=n-3$ allows us to turn the sum $1=1/2+1/3+1/6$ to the n-term sum

$$1=\frac{1}{2}+\frac{1}{6}+\frac{1}{n}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-1)n}.$$

This construction works provided that $n\neq k(k+1)$ for any k. Otherwise, we have $n\geq 6$, and instead we apply the above to the sum $1=1/2+1/3+1/10+1/15$, taking $k=3$, $r=n-4$ to yield

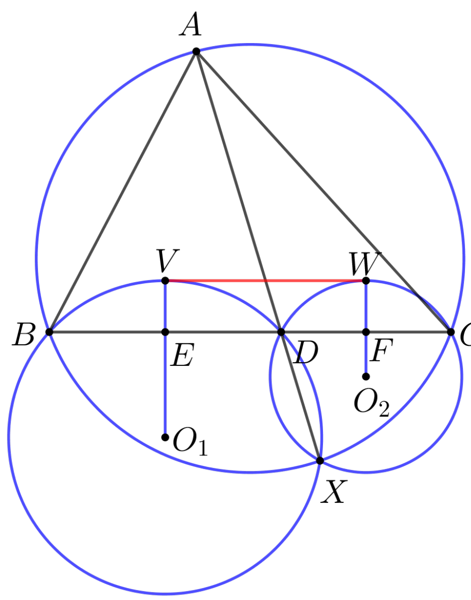
$$1=\frac{1}{2}+\frac{1}{10}+\frac{1}{15}+\frac{1}{n-1}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-2)(n-1)}.$$

The above then works, because if $n=k(k+1)$ for some k, then $n-1\neq 2,10,15$, and $n-1\neq m(m+1)$ for any m by parity, since $n-1$ is odd and $m(m+1)$ is always even.

This construction is not unique; there are other similar ones. □

6. In $\triangle ABC$, let D be the point on side BC such that $AB+BD=DC+CA$. The line AD intersects the circumcircle of $\triangle ABC$ again at point $X\neq A$. Prove that one of the common tangents of the circumcircles of $\triangle BDX$ and $\triangle CDX$ is parallel to BC.

Solution. Refer to the figure shown below:



Let V and W be the midpoints of arcs BD and CD respectively. We claim that VW is the desired common tangent. To prove this, let E and F be the orthogonal projections of V and W onto BC. Note that E and F are the midpoints of BD and CD respectively. Now we claim that $VE=WF$. To see this, note that

$$\begin{aligned}
VE &= BV\sin\angle VBE \\
&= \frac{BX\sin\angle BXV}{\sin\angle BDX}\sin\frac{\angle BXD}{2} \\
&= \frac{BX\sin\frac{\angle BXA}{2}}{\sin\angle BDX}\sin\frac{C}{2} \\
&= \frac{BX}{\sin\angle BDX}\sin^2\frac{C}{2}.
\end{aligned}$$

Similarly, we can prove that $WF = \frac{CX}{\sin \angle CDX} \sin^2 \frac{B}{2}$. Thus, to prove the claim, it suffices to prove that $\dfrac{BX}{CX} = \dfrac{\sin^2 \frac{B}{2}}{\sin^2 \frac{C}{2}}$. This is because

$$\frac{BX}{CX} = \frac{\sin \angle BAD}{\sin \angle CAD} = \frac{c/BD}{b/CD} = \frac{c(s-c)}{b(s-b)} = \frac{\sin^2 \frac{B}{2}}{\sin^2 \frac{C}{2}}.$$

This proves the first claim.

Next, we claim that VW is the desired common tangent. Note that from the first claim, $VWFE$ is a rectangle, since $\angle VEF$ and $\angle WFE$ are both right angles. Let O_1 and O_2 be the circumcenters of triangles BDX and CDX respectively. Then $O_1V \perp EF$, so since $EF||VW$ we get $O_1V \perp VW$. Likewise, $O_2W \perp VW$ as well, which proves the second claim.

It then follows that VW is the desired common tangent parallel to BC, and the required conclusion follows. □

7. Let a, b, and c be positive real numbers such that $ab + bc + ca = 3$. Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

Solution. It can be shown that

$$\frac{1}{1+a^4} \geq \frac{2-a^2}{2}.$$

Indeed, simplifying yields

$$\begin{aligned} 2 &\geq (1+a^4)(2-a^2) \\ (a^4+1)(a^2-2)+2 &\geq 0 \\ a^6 - 2a^4 + a^2 - 2 + 2 &\geq 0 \\ a^6 - 2a^4 + a^2 &\geq 0 \\ a^2(a^4 - 2a^2 + 1) &\geq 0 \\ a^2(a^2-1)^2 &\geq 0 \end{aligned}$$

which is true.

Thus

$$\begin{aligned} \frac{1}{1+a^4} &\geq \frac{2-a^2}{2} \\ \frac{bc}{1+a^4} &\geq \frac{2bc - a^2bc}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \frac{ca}{1+b^4} &\geq \frac{2ca - ab^2c}{2}, \text{ and} \\ \frac{ab}{1+c^4} &\geq \frac{2ab - abc^2}{2}. \end{aligned}$$

Adding these up yields

$$\frac{bc}{1+a^4}+\frac{ca}{1+b^4}+\frac{ab}{1+c^4}\geq(bc+ca+ab)-\frac{a^2bc+ab^2c+abc^2}{2}.$$

Observe that

$$\begin{aligned}(ab+bc+ca)^2&\geq 3((ab)(bc)+(bc)(ca)+(ca)(ab))\\ 9&\geq 3(a^2bc+ab^2c+abc^2)\\ 3&\geq a^2bc+ab^2c+abc^2.\end{aligned}$$

Thus,

$$(bc+ca+ab)-\frac{a^2bc+ab^2c+abc^2}{2}\geq 3-\frac{3}{2}=\frac{3}{2}.$$

Therefore,

$$\frac{bc}{1+a^4}+\frac{ca}{1+b^4}+\frac{ab}{1+c^4}\geq\frac{3}{2}.$$

□

8. The set $S=\{1,2,\ldots,2022\}$ is to be partitioned into n disjoint subsets $S_1,S_2,\ldots,S_n$ such that for each $i\in\{1,2,\ldots,n\}$, exactly one of the following statements is true:

(a) For all $x,y\in S_i$ with $x\neq y$, $\gcd(x,y)>1$.

(b) For all $x,y\in S_i$ with $x\neq y$, $\gcd(x,y)=1$.

Find the smallest value of n for which this is possible.

Solution. The answer is 15.

Note that there are 14 primes at most $\sqrt{2022}$, starting with 2 and ending with 43. Thus, the following partition works for 15 sets. Let $S_1=\{2,4,\ldots,2022\}$, the multiples of 2 in S. Let $S_2=\{3,9,15\ldots,2019\}$, the remaining multiples of 3 in S not in S_1. Let $S_3=\{5,25,35,\ldots,2015\}$ the remaining multiples of 5, and so on and so forth, until we get to $S_{14}=\{43,1849,2021\}$. S_{15} consists of the remaining elements, i.e. 1 and those numbers with no prime factors at most 43, i.e., the primes greater than 43 but less than 2022: $S_{15}=\{1,47,53,59,\ldots,2017\}$. Each of $S_1,S_2,\ldots,S_{14}$ satisfies i., while S_{15} satisfies ii.

We show now that no partition in 14 subsets is possible. Let a Type 1 subset of S be a subset S_i for which i. is true and there exists an integer $d>1$ for which d divides every element of S_i. Let a Type 2 subset of S be a subset S_i for which ii. is true. Finally, let a Type 3 subset of S be a subset S_i for which i. is true that is not a Type 1 subset. An example of a Type 3 subset would be a set of the form $\{pq,qr,pr\}$ where p,q,r are distinct primes.

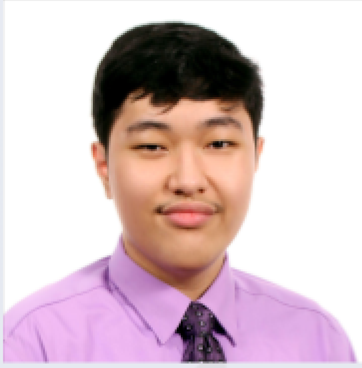
Claim: Let $p_1=2,p_2=3,p_3=5,\ldots$ be the sequence of prime numbers, where p_k is the kth prime. Every optimal partition of the set $S(k):=\{1,2,\ldots,p_k^2\}$, i.e., a partition with the least possible number of subsets, has at least $k-1$ Type 1 subsets. In particular, every optimal partition of this set has $k+1$ subsets in total. To see how this follows, we look at two cases:

- If every prime $p \leq p_k$ has a corresponding Type 1 subset containing its multiples, then a similar partitioning to the above works: Take S_1 to S_k as Type 1 subsets for each prime, and take S_{k+1} to be everything left over. S_{k+1} will never be empty, as it has 1 in it. While in fact it is known that, for example, by Bertrand's postulate there is always some prime between p_k and p_k^2 so S_{k+1} has at least two elements, there is no need to go this far–if there were no other primes you could just move 2 from S_1 into S_{k+1}, and if $k > 1$ then S_1 will still have at least three elements remaining. And if $k = 1$, there is no need to worry about this, because $2 < 3 < 2^2$.
- On the other hand, if $p \leq p_k$ has no corresponding Type 1 subset, then p and p^2 will not be contained in a Type 1 set. Neither can p nor p^2 be contained in a Type 3 set. If $\gcd(p, x) > 1$ for all x in the same set as p, then $\gcd(p, x) = p$, which implies that p is in a Type 1 set with $d = p$. Similarly, if $\gcd(p^2, x) > 1$ for all x in the same set as p^2, then $p \mid \gcd(p^2, x)$ for all x, and so p^2 is in a Type 1 set with $d = p$ as well. Hence p and p^2 must in fact be in Type 2 sets, and they cannot be in the same Type 2 set (as they share a common factor of $p > 1$); this means that the optimal partition has at least $k + 1$ subsets in total. A possible equality scenario for example is the sets $S_1 = \{1, 2, 3, 5, \ldots, p_k\}$, $S_2 = \{4, 9, 25, \ldots, p_k^2\}$, and S_3 to S_{k+1} Type 1 sets taking all remaining multiples of $2, 3, 5, \ldots, p_{k-1}$. This works, as p_k and p_k^2 are the only multiples of p_k in $S(k)$ with no prime factor other than p_k and thus cannot be classified into some other Type 1 set.

To prove our claim: We proceed by induction on k. Trivially, this is true for $k = 1$. Suppose now that any optimal partition of the set $S(k)$ has at least $k-1$ Type 1 subsets, and thus at least $k + 1$ subsets in total. Consider now a partition of the set $S(k + 1)$, and suppose that this partition would have at most $k + 1$ subsets. From the above, there exist at least two primes p, q with $p < q \leq p_{k+1}$ for which there are no Type 1 subsets. If $q < p_{k+1}$ we have a contradiction. Any such partition can be restricted to an optimal partition of $S(k)$ with $p < q \leq p_k$ having no corresponding Type 1 subsets. This contradicts our inductive hypothesis. On the other hand, suppose that $q = p_{k+1}$. Again restricting to $S(k)$ gives us an optimal partition of $S(k)$ with at most $k - 1$ Type 1 sets; the inductive hypothesis tells us that this partition has in fact exactly $k-1$ Type 1 sets and two Type 2 sets from a previous argument establishing the consequence of the claim. However, consider now the element pq. This cannot belong in any Type 1 set, neither can it belong in the same Type 2 set as p or p^2. Thus in addition to the given $k - 1$ Type 1 sets and 2 Type 2 sets, we need an extra set to contain pq. Thus our partition of $S(k + 1)$ in fact has at least $k - 1 + 2 + 1 = k + 2$ subsets, and not $k + 1$ subsets as we wanted. The claim is thus proved.

Returning to our original problem, since $p_{14} = 43 < \sqrt{2022}$, any partition of S must restrict to a partition of $S(14)$, which we showed must have at least 15 sets. Thus, we can do no better than 15. □

24th PMO Winners



Raphael Dylan T. Dalida
Philippine Science High School - Main Campus
CHAMPION

Filbert Ephraim S. Wu
Victory Christian International School
1ST RUNNER-UP

Jerome Austin N. Te
Jubilee Christian Academy
2ND RUNNER-UP

The prizes for the **top three contestants** in the National Stage are the following:

CHAMPION - Medal, trophy, certificate, Php 25,000
1ST RUNNER-UP - Medal, trophy, certificate, Php 20,000
2ND RUNNER-UP - Medal, trophy, certificate, Php 15,000

Their coaches will receive **Php 5,000, Php 3,000, and Php 2,000** respectively. Their schools will also receive a trophy.

Each of the special awardees will be given a medal and certificate. Their respective coaches will also receive a certificate.

24th PMO Highlights

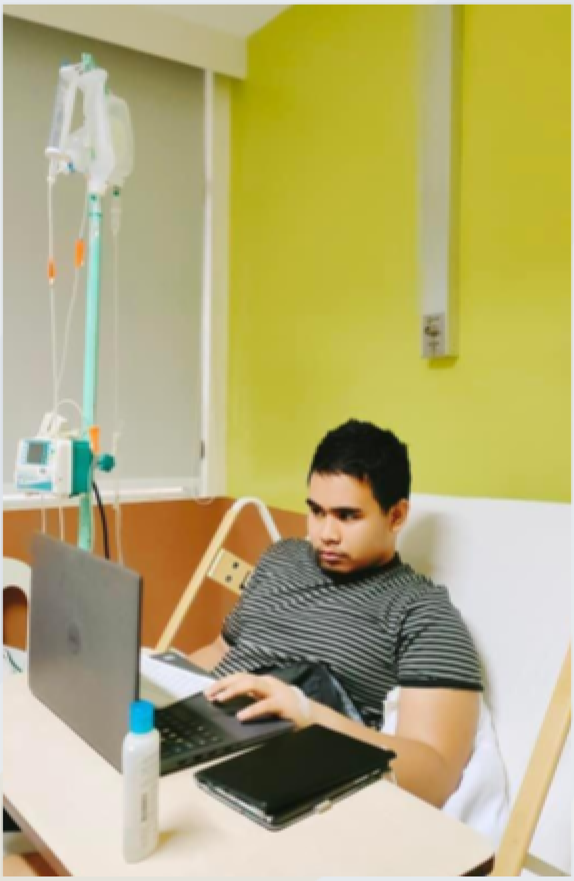
Erich Paredes of International School Manila (NCR) taking the PMO Qualifying Stage Exam

Students of Mainit National High School in Surigao del Norte (Region 13)
(credit: Mr. Val Jorge Stanley Basil)

Students of Ozamiz City National High School (Region 10) wearing their PMO shirts (credit: Mr. Nelson Tumala Jr.)

Students of Rawis National High School in Sorsogon City (Region 5) (credit: Mr. Michael Lusuegro)

Ramon Villota of Dayap National Integrated in Laguna (Region 4A)



Josh Robert Obaob of Ateneo de Davao University - Senior High School: in the hospital recovering from his appendicitis operation (Region 11)

24th PMO Highlights

Students of Cansilayan Farm School in Negros Occidental (Region 6)
(credit: Rachel Ricablanca)

Chalaice Cynt Matero of Vivencio P. Casas Sr. Memorial High School (Region 5)
(credit: Mr. Arjay Lequin)

24th PMO Highlights

The PMO National Stage winners with the MSP Board and DOST-SEI representatives during the online awarding ceremony on March 20, 2022, held through Zoom

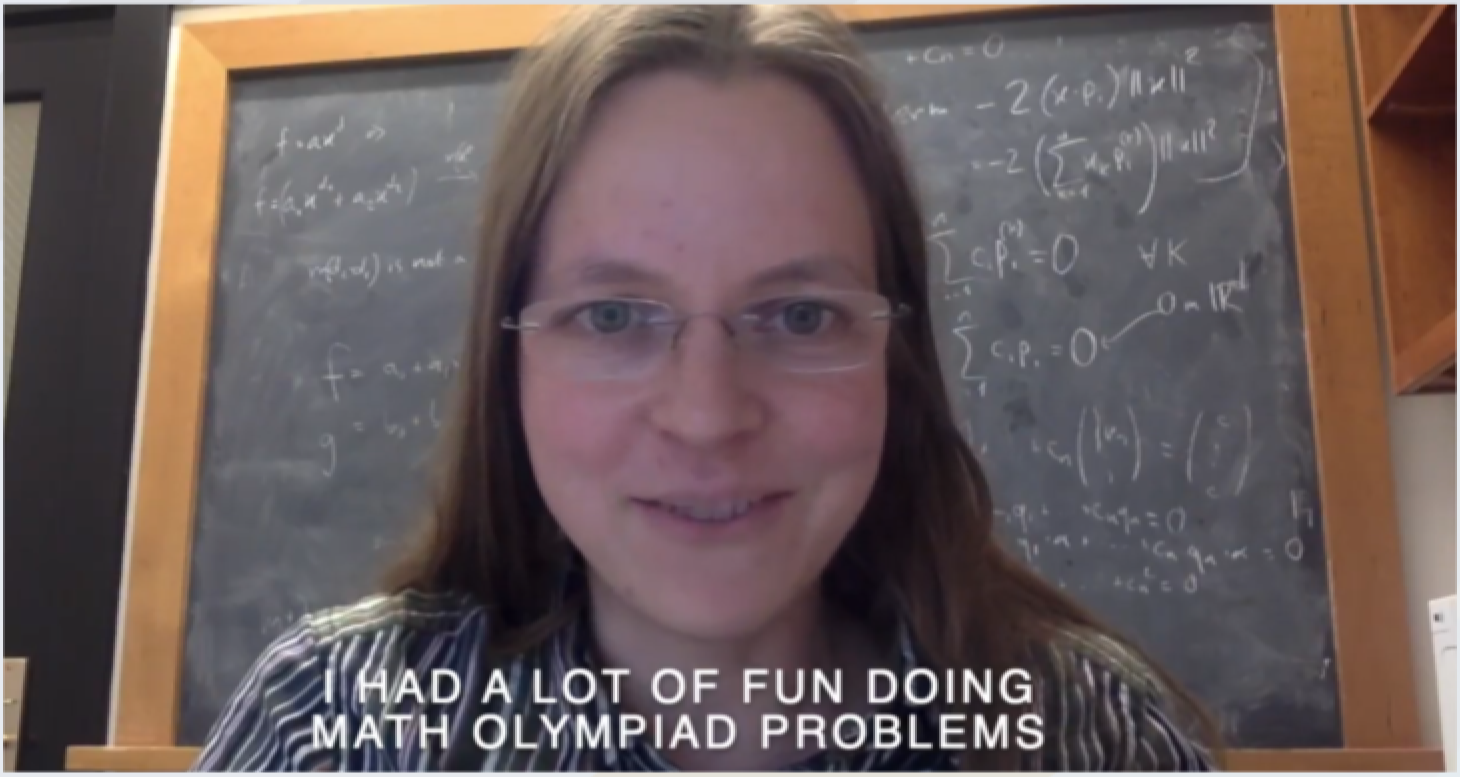
With the coaches of the National Stage Winners

24th PMO Highlights

The National Stage finalists with the MSP Board, DOST-SEI representatives, and PMO organizers.

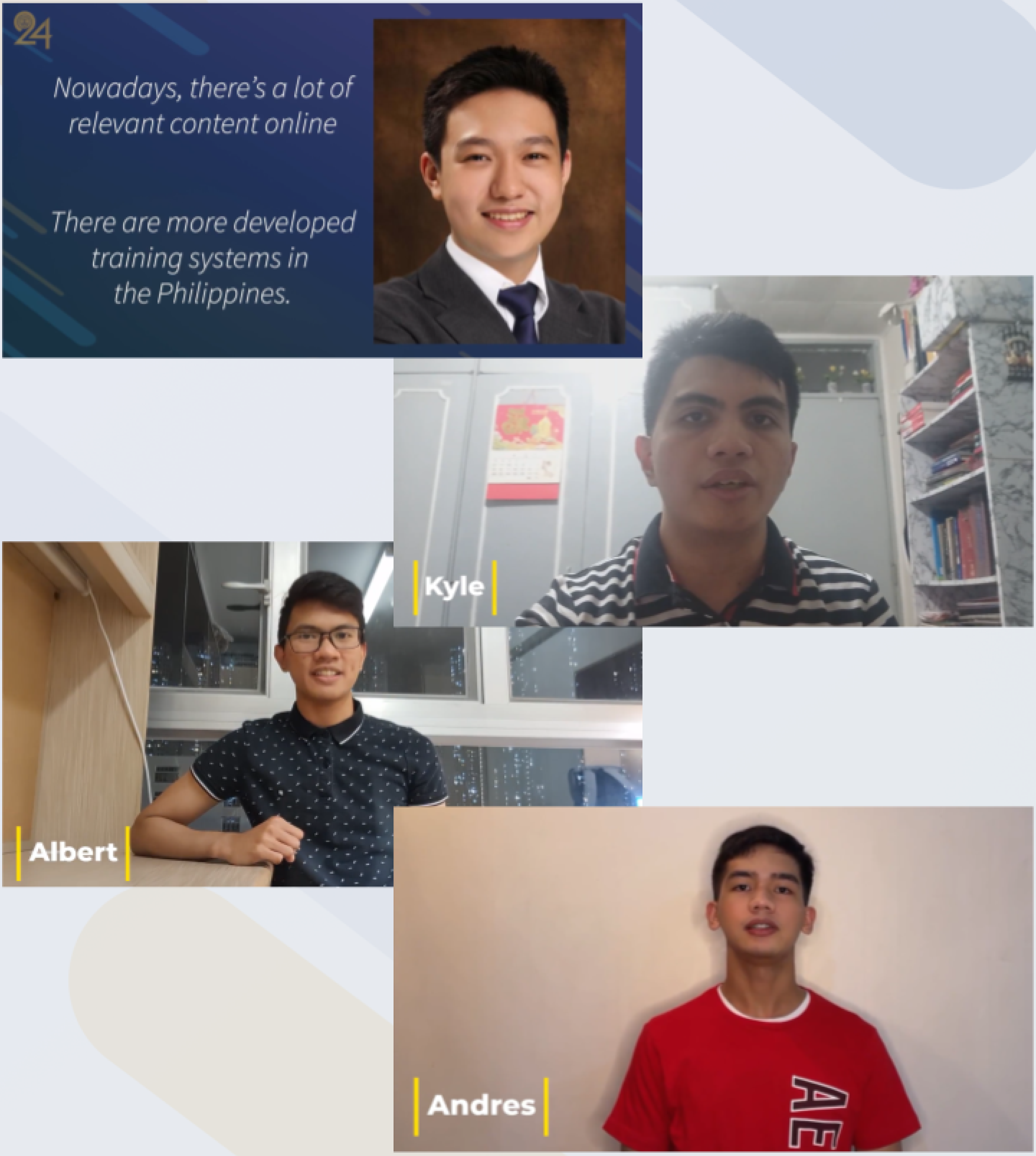
The Regional Coordinators of the PMO

Lisa Sauermann announced the Top Female Contestant in the Qualifying Stage. Dr. Sauermann currently teaches at MIT.



She represented Germany in the International Mathematical Olympiad, winning one silver medal and four gold medals. In her last IMO in 2011, she was the only one who got the perfect score.

24th PMO Highlights



The Philippines' four gold medalists in the IMO announced the Top Junior Contestants in the Qualifying Stage. They are (top to bottom) Farrell Eldrian Wu (IMO 2016 in Hong Kong), Kyle Patrick Dulay (IMO 2016 in Hong Kong), Albert John Patupat (IMO 2018 in Romania), and Andres Rico Gonzales III (IMO 2020 hosted by Russia).

ITEMHOUND

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We constantly innovate and improve our solutions,
enabling our partners to create more value to their captured audience.

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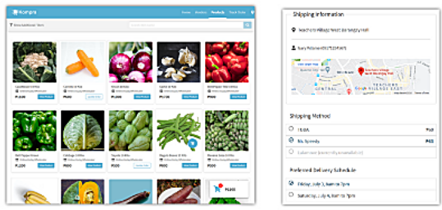
Virtual Race

Hold virtual races in the new normal. Promote health and wellness by encouraging people to participate at the comfort of their own place and time. Reward them with freebies delivered directly to their home.

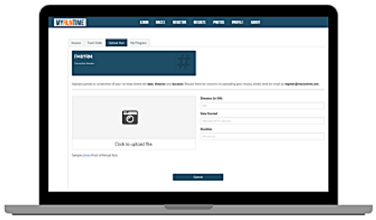
E-Commerce

Launch your own online store.
Provide options to your buyers to select their preferred delivery and payment methods. Take an option to have the items fulfilled by our team.



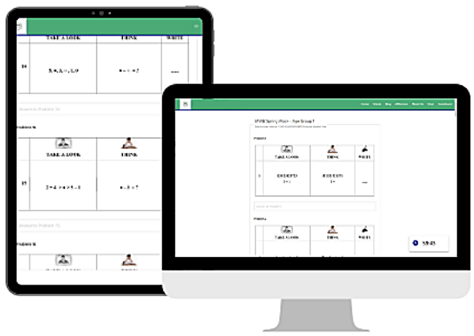
Online Registration

Create an online registration form for your event. Allow interested participants to securely submit their details and payment, with a peace of mind. Get access to see real-time conversions.



Online Exam

Allow your participants take their exams online. Prepare different types of questions and set the time limit for the exam. You can also set the start and end times to ensure that the exam is accessed only during the official schedule.



0917-568-6275
sales@itemhound.com
http://myrunti.me/
http://govirtual.goorahna.com/