

THE 24TH PHILIPPINE



24

MATHEMATICAL OLYMPIAD

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**QUALIFYING STAGE**

**February 19, 2022**

**NATIONAL STAGE**

**March 18-19, 2022**

**AWARDING CEREMONY**

**March 20, 2022**



**24**

**MATHEMATICAL OLYMPIAD**

# About the PMO

First held in **1984**, the **Philippine Mathematical Olympiad (PMO)** was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are:

- 1) to stimulate the improvement of mathematics education in the country by awakening greater interest in and appreciation of mathematics among students and teachers, and gaining insights into the levels of mathematical learning;
- 2) to identify and motivate the mathematically gifted;
- 3) to identify potential participants to the International Mathematical Olympiad;
- 4) to provide a vehicle for the professional growth of teachers; and
- 5) to encourage the involvement of both public and private sectors in the concerted promotion and development of mathematics education.

The PMO is only the first part of the selection program implemented by the Mathematical Society of the Philippines towards the country's participation in the **International Mathematical Olympiad (IMO)**. The thirty national finalists of the PMO will be invited to the **Mathematical Olympiad Summer Camp (MOSC)**, a training program where participants will experience problem solving at a level that will help them grow in mathematical maturity, in preparation for the IMO. The selection tests and quizzes will then determine the six contestants who will form the country's National Team in Mathematics - the Philippine Team to the International Mathematical Olympiad.

The Philippine Mathematical Olympiad, the Mathematical Olympiad Summer Camp, and the country's participation in the International Mathematical Olympiad, are projects of both the **Mathematical Society of the Philippines** and the **Department of Science and Technology - Science Education Institute**.

The PMO this year is the twenty-fourth since 1984. Because of the ongoing COVID-19 pandemic, this is the second year in a row that the PMO was **held online**. **Two thousand nine hundred fifteen (2,915)** contestants from Grades 7 to 12 participated in the Qualifying Stage last February 19, 2022, from which our **thirty-two national finalists**, along with other **special awardees**, were chosen. From the national finalists, the Philippine Team to the **63rd International Mathematical Olympiad** were chosen. The IMO this year will be held from **July 6 to 16** in **Oslo, Norway**. This year, the **IMO resumes onsite competitions** after being conducted online the previous two years.

# Message



**By Dr. Josette T. Biyo, Director, DOST-SEI**  
At the 24th Philippine Mathematical Olympiad

Greetings of safety and abundance!

The Department of Science and Technology – Science Education Institute takes pride in its partnership with the Mathematical Society of the Philippines in developing the mathematical minds of the Filipino youth through the Philippine Mathematical Olympiad.

The PMO, which is now in its 24<sup>th</sup> year, has become one of the most sought-after academic competitions for high school students in the country. Being the oldest and most prestigious nationwide mathematics competition, the PMO has honed and produced hundreds of talented mathematicians who are now making waves in the field of mathematics.

Last year, despite the challenges brought about by the pandemic, the PMO pushed through an online competition which was well-received by the contestants. The students, and their unending enthusiasm in competition and solving math problems, are our inspiration for continuing the work that we do in the Institute. The DOST-SEI will remain committed to its mandate to empower the Filipino youth to become scholars for the nation and be the best versions of themselves with meaningful contributions in the fields of Science, Technology, Engineering and Math.

It is important that we pursue the conduct of these kinds of competition and provide opportunities for our students to reach their maximum potential despite the changes the world is facing today. We must continue to help our students dream, compete, and win because the world is for them to conquer. Trust that DOST-SEI will be there to help students achieve their dreams to become the best in the field of STEM.

May this competition bring joy and excitement to all contestants and promote camaraderie amongst our students. Only through our collective efforts, expertise and intelligence will we be able to create a better future anchored in science and a development that is for all.

**JOSETTE T. BIYO**

Director, Science Education Institute  
Department of Science and Technology

# Message

**By Dr. Jose Ernie C. Lope, President, MSP**  
At the 24th Philippine Mathematical Olympiad

Congratulations to all the participants of the 24th edition of the Philippine Mathematical Olympiad, most especially to the national finalists and winners! Congratulations also to the PMO Organizing Team led by Dr. Richard Eden, all the Regional Coordinators, and the Test Development Committee headed by Dr. Christian Paul Chan Shio for the successful conduct of the Olympiad despite the challenges posed by the pandemic.

This year's set of finalists is particularly remarkable – more regions are now represented, and we have finalists coming from Luzon, Visayas, and Mindanao! Compared to the final rounds in recent memory, we also have many more females this year. We look forward to having more female finalists and more regional representatives in the years to come.

The Mathematical Society of the Philippines is committed to provide avenues for its members and for the larger Philippine mathematical community to pursue their research interests, disseminate their findings, and further enhance their mathematical skills. This commitment naturally includes the nurturing of young Filipinas and Filipinos, who we hope will pursue careers in mathematics and its allied fields.

What sets the PMO apart from the other math competitions is the quality of its questions – formulated by seasoned veterans and designed to identify potential members of the Philippine Team. The MSP is most certainly pleased with our Team's performance in the Asia Pacific and International Mathematical Olympiads in recent years. With the continued support of the DOST-SEI, we at the MSP are confident that our recent accomplishments will be sustained, if not surpassed.

*Padayon!*



**JOSE ERNIE C. LOPE**

President  
Mathematical Society of the Philippines



# The PMO Team

## Executive Committee

**Director** Dr. Richard B. Eden  
Department of Mathematics, Ateneo de Manila University

**Assistant Director** Dr. Ezra S. Aguilar  
Department of Mathematics and Physics, University of Santo Tomas

**Secretary** Dr. Daryl Allen B. Saddi  
Institute of Mathematics, University of the Philippines Diliman

## Operations Group

**Treasurer** Mr. Calvin S. Sia  
Ateneo de Manila University

**Project Assistant** Mrs. Gaudelia Ruiz  
Univeristy of the Philippines Diliman

**General Designer** Ms. Genesis Jacinth Tan  
Student, Ateneo de Manila University

**Webpage Designer** Mr. Daniel Del Rio

**Social Media  
Manager** Mr. Paul Vitus De Jesus  
University of Sto. Tomas

# The PMO Team

## Test Development

**Head** Dr. Christian Paul Chan Shio  
Ateneo de Manila University

**Critics** Dr. Louie John Vallejo  
University of the Philippines Diliman  
Mr. Carl Joshua Quines  
Student, Massachusetts Institute of Technology

**Members** Mr. Carlo Francisco Adajar  
Student, University of Georgia  
Mr. Kyle Patrick Dulay  
Student, Ateneo de Manila University  
Mr. Lu Christian Ong  
Ateneo de Manila University  
Mr. Gerard Francis Ortega  
Student, Ateneo de Manila University  
Mr. Sean Anderson Ty  
Student, Harvard University

## Regional Coordinators

**Region 1 & CAR** Mr. Anthony M. Pasion  
Department of Mathematics and Computer Science, University of the  
Philippines – Baguio

**Region 2** Mr. Crizaldy P. Binarao  
Department of Natural Sciences and Mathematics, Cagayan State  
University

**Region 3** Dr. Nancy L. Mati  
Mathematics Department, Tarlac State University

# The PMO Team

## Regional Coordinators

**Region 4A** Ms. Sharon P. Lubag

Mathematics and Statistics Department, De La Salle University – Dasmariñas

**Region 4B** Ms. Emmalyn T. Venturillo

College of Arts and Sciences, Western Philippines University

**Region 5** Ms. Glenda G. Quinto

Department of Mathematics, Ateneo de Naga University

**Region 6** Dr. Alexander J. Balsomo

Department of Mathematics, West Visayas State University

**Region 7** Dr. Cherrylyn P. Alota

Department of Mathematics and Statistics, University of the Philippines – Cebu

**Region 8** Mr. Oreste M. Ortega, Jr.

Mathematics Department, Leyte Normal University

**Region 9** Mr. Dante V. Partosa

Mathematics Department, Ateneo de Zamboanga University

**Region 10** Mr. Paolo B. Araune

Mathematics Department, Xavier University – Ateneo de Cagayan

**Region 11** Mr. Joseph T. Belida

Mathematics Department, Ateneo de Davao University

**Region 12, 13,  
& BARMM** Dr. Miraluna L. Herrera

Department of Mathematics, Caraga State University

**NCR** Dr. Kristine Joy E. Carpio

Mathematics and Statistics Department, De La Salle University

Dr. Jude C. Buot

Department of Mathematics, Ateneo de Manila University



# Special Awards

## Top Contestant Per Region from the Qualifying Stage

### **NCR**

Raphael Dylan T. Dalida  
Philippine Science High School -  
Main Campus

Jerome Austin N. Te  
Jubilee Christian Academy

### **CAR**

Han Na Park  
Philippine Science High School -  
Cordillera Administrative Region  
Campus

Karl Uriel Dela Cruz  
Philippine Science High School -  
Cordillera Administrative Region  
Campus

### **Region 1**

Phylline Cristel O. Calubayan  
BHC Educational Institution, Inc.

### **Region 2**

Grace B. Finotton  
Maddela Comprehensive High School

### **Region 3**

Justin Teng Soon T. Khoo  
Regional Science High School III

### **Region 4A**

Jan Neal Isaac D. Villamin  
Santa Rosa Science and  
Technology High School

### **Region 5**

Carl Dexter N. Donor  
Sorsogon National High School

John Angelo O. Oringo  
Philippine Science High School -  
Bicol Region Campus

### **Region 6**

Jonathan D. Anacan  
Philippine Science High School -  
Western Visayas Campus

# Special Awards

## Top Contestant Per Region from the Qualifying Stage

### Region 7

Lance Christopher M. Jimenez  
San Roque College de Cebu -  
Liloan Campus

### Region 11

Leonardo Florenz S. Eugenio  
Philippine Science High School -  
Southern Mindanao Campus

### Region 8

Evan Justin E. Panergo  
Calbayog City National High School

### Region 12

Felinwright Niñokyle A. Mesias  
Philippine Science High School -  
SOCCSKSARGEN Region Campus

### Region 9

Douglas Frederick S. Guangco  
Zamboanga Chong Hua High School

### Region 13

Matt Raymond Ayento  
Philippine Science High School -  
Caraga Region Campus

### Region 10

Mohammad Nur G. Casib  
Philippine Science High School -  
Central Mindanao Campus

### BARMM

Mohammad Miadh M. Angkad  
Albert Einstein School, Inc.

# Special Awards

## Top Contestant Per Area from the Qualifying Stage

### Luzon

**1st** Justin Teng Soon T. Khoo  
Regional Science High School III

**2nd** Jan Neal Isaac D. Villamin  
Santa Rosa Science and  
Technology High School

**3rd** Rainier M. Guinto  
Cavite Science Integrated School

**3rd** Justin M. Sarmiento  
Marcelo H. Del Pilar  
National High School

### Visayas

**1st** Jonathan D. Anacan  
Philippine Science High School -  
Western Visayas Campus

**2nd** Christopher James C. Yap  
St. John's Institute

**3rd** Evan Justin E. Panergo  
Calbayog City National High School

**3rd** Christian Jacob C. Yap  
St. John's Institute

### Mindanao

**1st** Mohammad Nur G. Casib  
Philippine Science High School -  
Central Mindanao Campus

**2nd** Mohammad Miadh M.  
Angkad  
Albert Einstein School, Inc.

**2nd** Matt Raymond Ayento  
Philippine Science High School - Caraga  
Region Campus

### NCR

**1st** Raphael Dylan T. Dalida  
Philippine Science High School -  
Main Campus

**1st** Jerome Austin N. Te  
Jubilee Christian Academy

**3rd** Luke Sebastian C. Sy  
Grace Christian College

**3rd** Shawn Darren S. Chua  
MGC New Life Christian Academy

**3rd** Alvann Walter W. Paredes Dy  
Saint Jude Catholic School

# Special Awards

## Top Female Contestant

Kristen Steffi S. Teh  
Ateneo de Manila  
Senior High School

## Top Junior Contestants

Mohammad Nur G. Casib  
Philippine Science High School -  
Central Mindanao Campus

Jerome Austin N. Te  
Jubilee Christian Academy

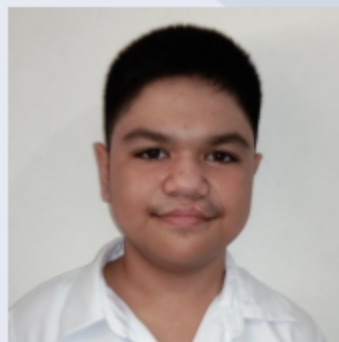
# National Finalists



**Karl Francois  
Dwayne N. Agudong**  
Quezon City Science  
High School



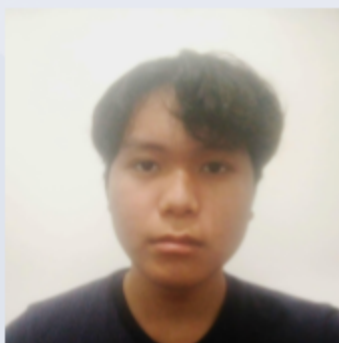
**Deanne Gabrielle D.  
Algenio**  
Ateneo de Manila  
Senior High School



**Jonathan D. Anacan**  
Philippine Science High  
School - Western Visayas  
Campus



**Mohammad Miadh  
M. Angkad**  
Albert Einstein School,  
Inc.



**Matt Raymond  
Ayento**  
Philippine Science High  
School - Caraga Region  
Campus



**Dominic Lawrence R.  
Bermudez**  
Philippine Science High  
School - Main Campus



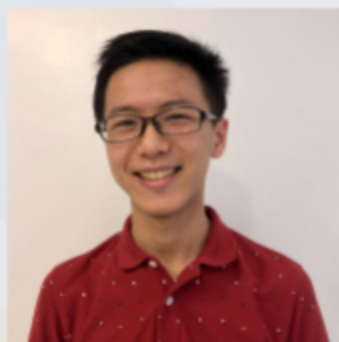
**Jose Maria S.  
Bernardo II**  
Ateneo de Manila  
Senior High School



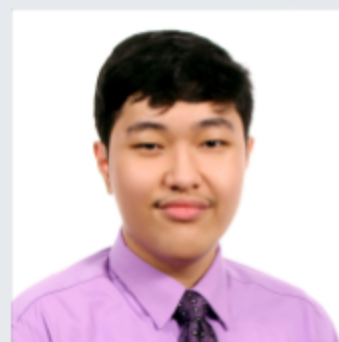
**Mohammad Nur G.  
Casib**  
Philippine Science High  
School - Central  
Mindanao Campus



**Enzo Rafael S. Chan**  
Ateneo de Manila  
Senior High School



**Shawn Darren S.  
Chua**  
MGC New Life Christian  
Academy



**Raphael Dylan T.  
Dalida**  
Philippine Science High  
School - Main Campus

# National Finalists



**Alexandra Brianne B.  
Gochian**  
Saint Jude Catholic  
School



**Benjamin L. Jacob**  
Philippine Science High  
School - Main Campus



**Justin Teng Soon T.  
Khoo**  
Regional Science High  
School III



**Hashanti Nikisha T.  
Liao**  
St. Stephen's High  
School



**Enrico Rolando G.  
Martinez**  
Philippine Science High  
School - Main Campus



**Martin Johan M.  
Ocho**  
De La Salle University  
Integrated School



**Citrei Kim K.  
Padayao**  
Quezon City Science  
High School



**Wesley Gavin G.  
Palomar**  
Philippine Science High  
School - Main Campus

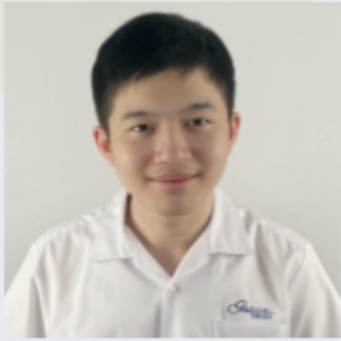


**Erich A. Paredes**  
International School  
Manila



**Alvann Walter W.  
Paredes Dy**  
Saint Jude Catholic  
School

# National Finalists



**Luke Sebastian C. Sy**  
Grace Christian College



**Cassidy Kyler L. Tan**  
Ateneo de Manila  
Senior High School



**Frederick Ivan T. Tan**  
Philippine Science High  
School - Main Campus



**Patricia Angelica P. Tan**  
De La Salle University  
Integrated School



**Rickson Caleb Y. Tan**  
MGC New Life Christian  
Academy



**Sean Matthew G. Tan**  
Jubilee Christian Academy



**Kean Nathaniel T. Tang**  
Ateneo de Manila  
Senior High School



**Jerome Austin N. Te**  
Jubilee Christian  
Academy



**Kristen Steffi S. Teh**  
Ateneo de Manila  
Senior High School



**Jan Neal Isaac D. Villamin**  
Santa Rosa Science and  
Technology High School



**Filbert Ephraim S. Wu**  
Victory Christian  
International School



# 24th Philippine Mathematical Olympiad

Qualifying Stage, 19 February 2022

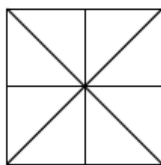
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**PART I.** Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

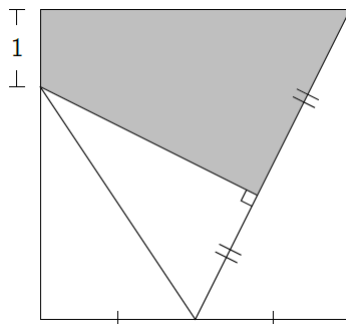
- Let  $XZ$  be a diameter of circle  $\omega$ . Let  $Y$  be a point on  $XZ$  such that  $XY = 7$  and  $YZ = 1$ . Let  $W$  be a point on  $\omega$  such that  $WY$  is perpendicular to  $XZ$ . What is the square of the length of the line segment  $WY$ ?  
(a) 7                      (b) 8                      (c) 10                      (d) 25
- How many five-digit numbers containing each of the digits 1, 2, 3, 4, 5 exactly once are divisible by 24?  
(a) 8                      (b) 10                      (c) 12                      (d) 20
- A lattice point is a point  $(x, y)$  where  $x$  and  $y$  are both integers. Find the number of lattice points that lie on the closed line segment whose endpoints are  $(2002, 2022)$  and  $(2022, 2202)$ .  
(a) 20                      (b) 21                      (c) 22                      (d) 23
- Let  $\omega \neq -1$  be a complex root of  $x^3 + 1 = 0$ . What is the value of  $1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^4$ ?  
(a) 3                      (b)  $-4$                       (c) 5                      (d)  $-6$
- How many ending zeroes does the decimal expansion of  $2022!$  have?  
(a) 404                      (b) 484                      (c) 500                      (d) 503
- Two tigers, Alice and Betty, run in the same direction around a circular track of circumference 400 meters. Alice runs at a speed of 10 m/s and Betty runs at 15 m/s. Betty gives Alice a 40 meter headstart before they both start running. After 15 minutes, how many times will they have passed each other?  
(a) 9                      (b) 10                      (c) 11                      (d) 12
- Suppose  $a, b, c$  are the roots of the polynomial  $x^3 + 2x^2 + 2$ . Let  $f$  be the unique monic polynomial whose roots are  $a^2, b^2, c^2$ . Find  $f(1)$ . (Note: A *monic* polynomial is a polynomial whose leading coefficient is 1.)  
(a)  $-17$                       (b)  $-16$                       (c)  $-15$                       (d)  $-14$
- Let  $I$  be the center of the incircle of triangle  $ABC$ . Suppose that this incircle has radius 3, and that  $AI = 5$ . If the area of the triangle is 2022, what is the length of  $BC$ ?  
(a) 670                      (b) 672                      (c) 1340                      (d) 1344



9. A square is divided into eight triangles as shown below. How many ways are there to shade exactly three of them so that no two shaded triangles share a common edge?



- (a) 12                      (b) 16                      (c) 24                      (d) 30
10. The numbers  $2, b, c, d, 72$  are listed in increasing order so that  $2, b, c$  form an arithmetic sequence,  $b, c, d$  form a geometric sequence, and  $c, d, 72$  form a harmonic sequence (that is, a sequence whose reciprocals of its terms form an arithmetic sequence). What is the value of  $b + c$ ?
- (a) 7                      (b) 13                      (c) 19                      (d) 25
11. How many positive integers  $n < 2022$  are there for which the sum of the odd positive divisors of  $n$  is 24?
- (a) 7                      (b) 8                      (c) 14                      (d) 15
12. Call a whole number *ordinary* if the product of its digits is less than or equal to the sum of its digits. How many numbers from the set  $\{1, 2, \dots, 999\}$  are ordinary?
- (a) 151                      (b) 162                      (c) 230                      (d) 241
13. What is the area of the shaded region of the square below?



- (a) 7                      (b) 11                      (c) 15                      (d) 19
14. Bryce plays a game in which he flips a fair coin repeatedly. In each flip, he obtains two tokens if the coin lands on heads, and loses one token if the coin lands on tails. At the start, Bryce has nine tokens. If after nine flips, he also ends up with nine tokens, what is the probability that Bryce always had at least nine tokens?
- (a)  $1/7$                       (b)  $1/6$                       (c)  $5/28$                       (d)  $17/84$

15. How many ways are there to arrange the first ten positive integers such that the multiples of 2 appear in increasing order, and the multiples of 3 appear in decreasing order?
- (a) 720                      (b) 2160                      (c) 5040                      (d) 6480

**PART II.** All answers are positive integers. Do not use commas if there are more than 3 digits, e.g., type 1234 instead of 1,234. A fraction  $a/b$  is in lowest terms if  $a$  and  $b$  are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. What is the largest multiple of 7 less than 10,000 which can be expressed as the sum of squares of three consecutive numbers?
17. Suppose that the polynomial  $P(x) = x^3 + 4x^2 + bx + c$  has a single root  $r$  and a double root  $s$  for some distinct real numbers  $r$  and  $s$ . Given that  $P(-2s) = 324$ , what is the sum of all possible values of  $|c|$ ?
18. Let  $m$  and  $n$  be relatively prime positive integers. If  $m^3n^5$  has 209 positive divisors, then how many positive divisors does  $m^5n^3$  have?
19. Let  $x$  be a positive real number. What is the maximum value of  $\frac{2022x^2 \log(x + 2022)}{(\log(x + 2022))^3 + 2x^3}$ ?
20. Let  $a, b, c$  be real numbers such that

$$3ab + 2 = 6b, \quad 3bc + 2 = 5c, \quad 3ca + 2 = 4a.$$

Suppose the only possible values for the product  $abc$  are  $r/s$  and  $t/u$ , where  $r/s$  and  $t/u$  are both fractions in lowest terms. Find  $r + s + t + u$ .

21. You roll a fair 12-sided die repeatedly. The probability that all the primes show up at least once before seeing any of the other numbers can be expressed as a fraction  $p/q$  in lowest terms. What is  $p + q$ ?
22. Let  $PMO$  be a triangle with  $PM = 2$  and  $\angle PMO = 120^\circ$ . Let  $B$  be a point on  $PO$  such that  $PM$  is perpendicular to  $MB$ , and suppose that  $PM = BO$ . The product of the lengths of the sides of the triangle can be expressed in the form  $a + b\sqrt[3]{c}$ , where  $a, b, c$  are positive integers, and  $c$  is minimized. Find  $a + b + c$ .
23. Let  $ABC$  be a triangle such that the altitude from  $A$ , the median from  $B$ , and the internal angle bisector from  $C$  meet at a single point. If  $BC = 10$  and  $CA = 15$ , find  $AB^2$ .
24. Find the sum of all positive integers  $n$ ,  $1 \leq n \leq 5000$ , for which

$$n^2 + 2475n + 2454 + (-1)^n$$

is divisible by 2477. (Note that 2477 is a prime number.)

25. For a real number  $x$ , let  $[x]$  denote the greatest integer not exceeding  $x$ . Consider the function

$$f(x, y) = \sqrt{M(M+1)} \left( |x - m| + |y - m| \right),$$

where  $M = \max([x], [y])$  and  $m = \min([x], [y])$ . The set of all real numbers  $(x, y)$  such that  $2 \leq x, y \leq 2022$  and  $f(x, y) \leq 2$  can be expressed as a finite union of disjoint regions in the plane. The sum of the areas of these regions can be expressed as a fraction  $a/b$  in lowest terms. What is the value of  $a + b$ ?

## Answers

### Part I. (2 points each)

- |      |       |       |
|------|-------|-------|
| 1. A | 6. D  | 11. D |
| 2. B | 7. C  | 12. D |
| 3. B | 8. A  | 13. D |
| 4. D | 9. B  | 14. A |
| 5. D | 10. C | 15. D |

### Part II. (5 points each)

16.  $\boxed{8750}$

Let the number be expressed as  $a^2 + (a + 1)^2 + (a + 2)^2$ , where  $a$  is an integer. It may be checked that this expression is a multiple of 7 if and only if the remainder when  $a$  is divided by 7 is 1 or 4. In the former case, the largest possible value of  $a$  that places the value of the expression within bounds is 50, which gives the value  $50^2 + 51^2 + 52^2 = 7805$ . In the latter case, the largest such value of  $a$  is 53, which gives the value  $53^2 + 54^2 + 55^2 = 8750$ .

17.  $\boxed{108}$

By Vieta's formula, we have  $r + 2s = -4$  and writing  $P(x) = (x - r)(x - s)^2$ , we have  $324 = P(-2s) = (-2s - r)(-3s)^2 = 36s^2$ . Thus,  $s^2 = 9$  and  $s \in \{-3, 3\}$ . We next observe that  $|c| = |P(0)| = |r|s^2 = |-4 - 2s|s^2$ . Hence, the sum of all possible values of  $|c|$  is  $9(|-4 - 6| + |-4 + 6|) = 108$ .

18.  $\boxed{217}$

Let  $d(N)$  denote the number of positive divisors of an integer  $N$ . Suppose that the prime factorizations of  $m$  and  $n$  are  $\prod(p_i^{a_i})$  and  $\prod(q_i^{b_i})$  respectively. Observe that 209 has four positive divisors: 1, 11, 19, 209. If  $m = 1$ , then  $n^5$  would have 209 divisors. Thus,  $d(n^5) = 209$ . However, it is known that  $d(n^5) = \prod(5b_i + 1)$ , but the latter implies that the remainder of  $d(n^5)$  when divided by 5 is 1, a contradiction.

Likewise, if  $n = 1$ , then it implies that the remainder when  $d(m^3) = 209$  is divided by 3 is 1, also a contradiction.

Thus,  $m, n > 1$ , so  $d(m), d(n) > 1$ . As  $m$  and  $n$  are relatively prime, then  $d(m^3n^5) = d(m^3)d(n^5)$ . As the only way to factor 209 as a product of 2 integers greater than 1 is  $11 \cdot 19$ , then  $d(m^3)$  and  $d(n^5)$  are 11 and 19 in some order. As the remainder when  $d(m^3)$  and  $d(n^5)$  is divided by 3 and 5 respectively is 1, then  $d(m^3) = 19 = (3 \cdot 6 + 1)$ , and  $d(n^5) = 11 = (5 \cdot 2 + 1)$ . Thus,  $m$  and  $n$  can be expressed as  $p^6$  and  $q^2$  respectively. Therefore,  $d(m^5n^3) = d(p^{30}q^6) = (30 + 1)(6 + 1) = 217$ .

19.  $\boxed{674}$

By the AM-GM Inequality,

$$\begin{aligned} \frac{2022x^2 \log(x + 2022)}{\log^3(x + 2022) + 2x^3} &= \frac{2022x^2 \log(x + 2022)}{\log^3(x + 2022) + x^3 + x^3} \\ &\leq \frac{2022x^2 \log(x + 2022)}{3(x^6 \log^3(x + 2022))^{\frac{1}{3}}} \\ &= \frac{2022x^2 \log(x + 2022)}{3x^2 \log(x + 2022)} \\ &= 674 \end{aligned}$$

Equality holds when  $x = \log(x + 2022)$ , which has a positive solution.

20. 18

The three given equations can be written as

$$3a + \frac{2}{b} = 12, \quad 3b + \frac{2}{c} = 10, \quad 3c + \frac{2}{a} = 8.$$

The product of all the three equations gives us

$$27abc + 6 \left(3a + \frac{2}{b}\right) + 6 \left(3b + \frac{2}{c}\right) + 6 \left(3c + \frac{2}{a}\right) + \frac{8}{abc} = 120.$$

Plugging in the values and simplifying the equation gives us

$$27(abc)^2 - 30abc + 8 = 0.$$

This gives  $abc$  as either  $4/9$  or  $2/3$ , so  $r + s + t + u = 4 + 9 + 2 + 3 = 18$ .

21. 793

There are 5 primes which are at most 12 – namely 2,3,5,7 and 11. Notice that if a number has already been seen, we can effectively ignore all future occurrences of the number. Thus, the desired probability is the fraction of the permutations of  $(1, 2, \dots, 12)$  such that the primes all occur first. There are  $5!$  ways to arrange the primes, and  $7!$  ways to arrange the composites to satisfy the condition. Since there are  $12!$  possible permutations, the desired probability is  $\frac{5!7!}{12!} = \frac{1}{792}$ , and the required sum is  $1 + 792 = 793$ .

22. 28

Extend  $PM$  to a point  $C$  such that  $PC \perp OC$ . Since  $\angle PMO = 120^\circ$ ,  $\angle CMO = 60^\circ$  and  $\angle COM = 30^\circ$ . Let  $PB = x$  and  $MC = a$ . Then  $CO = a\sqrt{3}$  and  $OM = 2a$ . Moreover,  $\triangle PMB$  and  $\triangle PCO$  are similar triangles. Thus, we have

$$\frac{2}{2+a} = \frac{x}{x+2}$$

so  $x = 4/a$ .

Furthermore, by the Cosine Law on side  $PO$  of  $\triangle PMO$ , we have

$$(x+2)^2 = 4 + 4a^2 + 2a$$

Plugging in  $x = 4/a$  and expanding, we have

$$\frac{16}{a^2} + \frac{16}{a} + 4 = 4 + 4a^2 + 4a$$

and so  $4 + 4a = a^4 + a^3$ . Hence  $a^3 = 4$  and  $a = \sqrt[3]{4}$ . Thus,  $x = 4/\sqrt[3]{4}$ .

It follows that the product of the lengths of the sides of the triangle is

$$(2a)(2)(x+2) = (2\sqrt[3]{4})(2)(2\sqrt[3]{2} + 2) = 16 + 8\sqrt[3]{4},$$

so  $a + b + c = 16 + 8 + 4 = 28$ .

23. 205

Let  $D$  be the foot of the  $A$ -altitude,  $E$  the midpoint of  $AC$ , and  $F$  the foot of the  $C$ -internal angle bisector. Then by Ceva's Theorem, we have  $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ , and so  $\frac{b}{a} \cdot \frac{c \cos B}{b \cos C} = 1$ , where we are using the shorthand  $BC = a, CA = b, AB = c$ . By cosine law, we know that  $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$  and  $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ . Substituting this into the equation, we obtain

$$a(a^2 + b^2 - c^2) = b(c^2 + a^2 - b^2)$$

Solving for  $c^2 = AB^2$  in this equation then gives the final answer, which is 205.

24. 9912

The problem is equivalent to finding all values of  $n$  that satisfies the following congruence:

$$n^2 - 2n - 23 + (-1)^n \equiv 0 \pmod{2477}.$$

If  $n$  is odd, then  $(-1)^n = -1$ . Then, we have:

$$n^2 - 2n - 24 \equiv 0 \pmod{2477}$$

$$(n - 6)(n + 4) \equiv 0 \pmod{2477}.$$

This gives us solutions  $n \equiv 6$  and  $n \equiv -4$  (modulo 2477). Since 2477 is prime, these are the only roots modulo 2477. This yields the solutions 6, 2483, 2473 and 4950. We only accept the odd values, 2483 and 2473.

If  $n$  is even, then  $(-1)^n = 1$ . Then, we have

$$n^2 - 2n - 22 \equiv 0 \pmod{2477}.$$

This gives  $n \equiv 1 \pm \sqrt{23} \pmod{2477}$ . Again, because 2477 is prime, these two are the only roots, modulo 2477. Since  $2477 + 23 = 2500 = 50^2$ , note that  $\sqrt{23} \equiv 50 \pmod{2477}$ , which then gives the solutions  $n \equiv 51$  and  $n \equiv -49$  (modulo 2477), yielding us the solutions 51, 2528, 2428 and 4905. We only accept the even values, 2528 and 2428.

Thus, the sum of all the solutions is  $2483 + 2473 + 2528 + 2428 = 9912$ .

25. 2021

Fix  $m \geq 2$ . First note that  $x, y \geq m$ , and that  $M \geq 2$ , implying  $\sqrt{M(M+1)} \geq 2$ . Thus,  $f(x, y) \leq 2$  necessarily implies  $|x - m| + |y - m| \leq 1$ , and so  $|x - m|, |y - m| \leq 1$ . This implies  $x \in [m - 1, m + 1]$ , and so  $x \in [m, m + 1]$ . Similarly,  $y \in [m, m + 1]$ . Now if  $x = m + 1$ , then  $1 \leq |x - m| + |y - m| \leq \frac{2}{\sqrt{M(M+1)}} < 1$ , contradiction. Using the same argument for  $y$ , it follows that  $x, y \in [m, m + 1)$ . Thus,  $\lfloor x \rfloor = \lfloor y \rfloor = m$  always, and so  $M = m$ .

The inequality is then equivalent to  $|x - m| + |y - m| \leq \frac{2}{\sqrt{m(m+1)}}$ . Let  $r \leq 1$  be the upper bound in the inequality. Keeping in mind the fact that  $x, y \geq m$ , this region is a right-triangle with vertices  $(m, m)$ ,  $(m + r, m)$  and  $(m, m + r)$ , which then has area  $\frac{r^2}{2} = \frac{2}{m(m+1)}$ . This region is within the set for  $2 \leq m \leq 2021$ , so the desired sum is

$$\sum_{m=2}^{2021} \frac{2}{m(m+1)} = 2 \sum_{m=2}^{2021} \left( \frac{1}{m} - \frac{1}{m+1} \right) = 2 \left( \frac{1}{2} - \frac{1}{2022} \right) = \frac{1010}{1011}.$$

It then follows that  $a + b = 2021$ .



# 24<sup>th</sup> Philippine Mathematical Olympiad

National Stage (Day 1)

18 March 2022

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Time: 4.5 hours

Each item is worth 7 points.

1. Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers  $a, b, c,$  and  $d.$

2. The PMO Magician has a special party game. There are  $n$  chairs, labelled 1 to  $n.$  There are  $n$  sheets of paper, labelled 1 to  $n.$
- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
  - She then asks  $n$  party guests to sit on the chairs so that each chair has exactly one occupant.
  - Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if  $1 < m \leq n,$  where  $m$  is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly  $m$  claps.

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer  $k,$  denote by  $S_k$  the set of all visible lattice points  $(x, y)$  such that  $x^2 + y^2 = k^2.$  Let  $D$  denote the set of all positive divisors of  $2021 \cdot 2025.$  Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point  $(x, y)$  on the plane where both  $x$  and  $y$  are integers, and  $|A|$  denotes the number of elements of the set  $A.$

4. Let  $\triangle ABC$  have incenter  $I$  and centroid  $G.$  Suppose that  $P_A$  is the foot of the perpendicular from  $C$  to the exterior angle bisector of  $B,$  and  $Q_A$  is the foot of the perpendicular from  $B$  to the exterior angle bisector of  $C.$  Define  $P_B, P_C, Q_B,$  and  $Q_C$  similarly. Show that  $P_A, P_B, P_C, Q_A, Q_B,$  and  $Q_C$  lie on a circle whose center is on line  $IG.$



# 24<sup>th</sup> Philippine Mathematical Olympiad

*National Stage (Day 2)*

19 March 2022

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*Time: 4.5 hours*

*Each item is worth 7 points.*

5. Find all positive integers  $n$  for which there exists a set of exactly  $n$  distinct positive integers, none of which exceed  $n^2$ , whose reciprocals add up to 1.

6. In  $\triangle ABC$ , let  $D$  be the point on side  $BC$  such that  $AB + BD = DC + CA$ . The line  $AD$  intersects the circumcircle of  $\triangle ABC$  again at point  $X \neq A$ . Prove that one of the common tangents of the circumcircles of  $\triangle BD X$  and  $\triangle CD X$  is parallel to  $BC$ .

7. Let  $a, b$ , and  $c$  be positive real numbers such that  $ab + bc + ca = 3$ . Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

8. The set  $S = \{1, 2, \dots, 2022\}$  is to be partitioned into  $n$  disjoint subsets  $S_1, S_2, \dots, S_n$  such that for each  $i \in \{1, 2, \dots, n\}$ , exactly one of the following statements is true:

(a) For all  $x, y \in S_i$  with  $x \neq y$ ,  $\gcd(x, y) > 1$ .

(b) For all  $x, y \in S_i$  with  $x \neq y$ ,  $\gcd(x, y) = 1$ .

Find the smallest value of  $n$  for which this is possible.



# 24<sup>th</sup> Philippine Mathematical Olympiad

## National Stage (Solutions)

18-19 March 2022

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1. Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers  $a, b, c,$  and  $d$ .

Solution. We claim that the only solutions are  $f(x) = 0$  and  $f(x) = x$ . It is easy to see that both satisfy the functional inequality. We now show that no other functions satisfy the given condition.

Letting  $a = b = c = d$  yields  $2f(0)^2 \leq 0$  which implies  $f(0) = 0$ . Now, suppose that  $f \not\equiv 0$ . Then we can find  $p \in \mathbb{R}$  such that  $f(p) \neq 0$ . For  $a \in \mathbb{R}$ , let  $b = a, c = 0,$  and  $d = a - p$ . We then have

$$f(p)f(a) \leq af(p).$$

*Case 1:* Suppose  $f(p) > 0$ . Then  $f(a) \leq a$  for all  $a \in \mathbb{R}$ . In particular  $f(-1) < 0$ . In the original functional equation, letting  $b = a, c = 0,$  and  $d = a + 1$  yields  $f(-1)f(a) \leq af(-1)$  for all  $a \in \mathbb{R}$ . Hence,  $f(a) \geq a$  for all  $a \in \mathbb{R}$ . Thus for this case,  $f(x) = x$  for all  $x \in \mathbb{R}$ .

*Case 2:* Suppose  $f(p) < 0$ . Then  $f(a) \geq a$  for all  $a \in \mathbb{R}$ . In particular  $f(1) > 0$ . In the original functional equation, letting  $b = a, c = 0,$  and  $d = a - 1$  yields  $f(1)f(a) \leq af(1)$  for all  $a \in \mathbb{R}$ . Hence,  $f(a) \leq a$  for all  $a \in \mathbb{R}$ . Thus for this case,  $f(x) = x$  for all  $x \in \mathbb{R}$ . □

2. The PMO Magician has a special party game. There are  $n$  chairs, labelled 1 to  $n$ . There are  $n$  sheets of paper, labelled 1 to  $n$ .

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks  $n$  party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if  $1 < m \leq n$ , where  $m$  is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly  $m$  claps.

Solution. Decompose the permutation into cycles of lengths  $c_1, c_2, c_3, \dots, c_k$ . Note that  $c_1 + c_2 + \dots + c_k = n$ . A guest in a cycle of length  $c$  returns to their original seat after  $c$  claps. Thus, all guests return to their original seats after  $\text{lcm}\{c_i\}$  claps.



Let  $m = pq$ , where  $p$  and  $q$  are coprime and both greater than 1; if  $m > 1$  is not a prime power, then it has at least two different prime factors, so this is always possible. Suppose we have at least one cycle of lengths  $p$  and  $q$ , and *all* cycles are of lengths  $p$  or  $q$ . Then, all guests will return to their original seats after  $\text{lcm}\{p, q\} = m$  claps.

If  $m = n$ , we can just connect all  $n$  party guests in one big cycle of size  $m = n$ . Otherwise,  $m < n$ , and note that the problem is equivalent to finding a nonnegative integer solution to  $p\bar{x} + q\bar{y} = n - p - q$ . Since  $m < n$ , a solution always exists by the Chicken McNugget Theorem, completing the proof.  $\square$

3. Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer  $k$ , denote by  $S_k$  the set of all visible lattice points  $(x, y)$  such that  $x^2 + y^2 = k^2$ . Let  $D$  denote the set of all positive divisors of  $2021 \cdot 2025$ . Compute the sum

$$\sum_{d \in D} |S_d|.$$

Here, a lattice point is a point  $(x, y)$  on the plane where both  $x$  and  $y$  are integers, and  $|A|$  denotes the number of elements of the set  $A$ .

Solution. We claim that the required sum is 20.

Let  $T_k$  denote the set of all lattice points in the circle  $x^2 + y^2 = k^2$ . We claim that  $\sum_{d|k} |S_d| = |T_k|$ . Indeed, given a point  $(x, y)$  in  $T_k$ , let  $g = \text{gcd}(x, y)$ . Then  $x/g, y/g$  are necessarily coprime, and hence  $(x/g, y/g)$  visible, and  $(x/g)^2 + (y/g)^2 = (k/g)^2$ . This implies  $(x/g, y/g) \in \bigcup_{d|k} S_d$ . Next, note that the  $S_d$ 's are necessarily disjoint. Now if  $(x', y')$  is a visible lattice point in  $S_d$  where  $d|k$ , then we can write  $k = gd$  so that  $(x, y) = (gx', gy')$  is a lattice point in  $T_k$ . This establishes a bijection between  $\bigcup_{d|k} S_d$  and  $T_k$ , and since the  $S_d$ 's are disjoint, the claim follows.

From the claim, it suffices to find the number of lattice points in the circle  $x^2 + y^2 = (2021 \cdot 2025)^2$ . This is equivalent to

$$x^2 + y^2 = 3^8 \cdot 5^4 \cdot 43^2 \cdot 47^2.$$

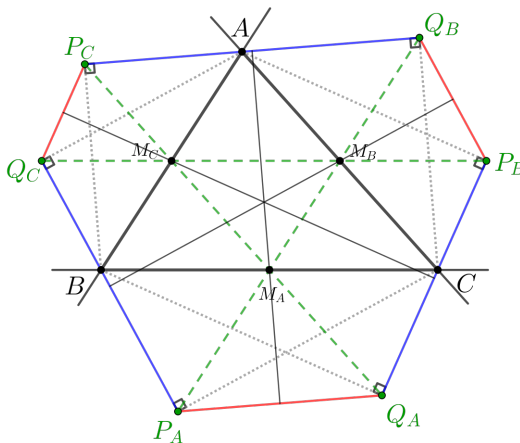
Now it is well-known that if  $x^2 + y^2 \equiv 0 \pmod{p}$  where  $p \equiv 3 \pmod{4}$  is a prime, then  $x \equiv y \equiv 0 \pmod{p}$ . Thus, we must also have  $x, y \equiv 0 \pmod{3^4 \cdot 43 \cdot 47}$ . It then follows that the number of lattice points is the same as the number of lattice points in  $x^2 + y^2 = 25^2$ .

If  $x = 0$  or  $y = 0$ , there are 4 solutions. Otherwise, assume WLOG that they are both positive. Now it is well-known that all solutions to  $x^2 + y^2 = z^2$  are in the form  $x = g(m^2 - n^2)$ ,  $y = 2gmn$ , and  $z = g(m^2 + n^2)$ , where  $m > n$  are coprime positive integers, and  $g$  is a positive integer. Thus, we want  $g(m^2 + n^2) = 25$ . Note that  $g|25$ , so  $g = 1, 5, 25$ .

If  $g = 25$ , then  $m^2 + n^2 = 1$ , so  $n = 0$ , contradiction. If  $g = 5$ , then  $m^2 + n^2 = 5$ , which yields  $m = 2$  and  $n = 1$  and thus  $g(m^2 - n^2) = 15$  and  $2gmn = 20$ , so  $(x, y) = (15, 20), (20, 15)$ . If  $g = 1$ , then we get  $m^2 + n^2 = 25$ , from which we obtain  $m = 4$  and  $n = 3$ . It then follows that  $(x, y) = (24, 7), (7, 24)$ , and so we have 2 solutions when  $x, y$  are both positive. This implies that there are  $4 \cdot 4 = 16$  solutions when  $x, y$  are nonzero, and so there are  $4 + 16 = 20$  solutions in total.  $\square$

4. Let  $\triangle ABC$  have incenter  $I$  and centroid  $G$ . Suppose that  $P_A$  is the foot of the perpendicular from  $C$  to the exterior angle bisector of  $B$ , and  $Q_A$  is the foot of the perpendicular from  $B$  to the exterior angle bisector of  $C$ . Define  $P_B, P_C, Q_B,$  and  $Q_C$  similarly. Show that  $P_A, P_B, P_C, Q_A, Q_B,$  and  $Q_C$  lie on a circle whose center is on line  $IG$ .

Solution. Refer to the figure shown below:



Let  $M_A, M_B,$  and  $M_C$  be the midpoints of  $BC, CA,$  and  $AB$  respectively.

First, it may be shown that  $P_B$  and  $Q_C$  lie on  $M_B M_C$ .

Note that  $\angle AM_C Q_C = 2\angle ABQ_C = 180^\circ - \angle ABC = \angle BM_C M_B$ . Thus,  $Q_C$  lies on  $M_B M_C$ . Likewise,  $\angle AM_B P_B = 2\angle ACP_B = 180^\circ - \angle ACB = \angle CM_B M_C$ . Thus,  $P_B$  also lies on  $M_B M_C$ .

Similarly,  $P_C$  and  $Q_A$  lie on  $M_C M_A$ , and  $P_A$  and  $Q_B$  lie on  $M_A M_B$ .

Now, as  $M_A$  is the center of the circle passing through  $B, C, Q_A,$  and  $P_A$ , then  $M_A P_A = M_A Q_A$ , so the angle bisector of  $\angle M_C M_A M_B$  coincides with the perpendicular bisector of  $P_A Q_A$ .

Observe that  $M_C Q_A = M_C M_A + M_A Q_A = \frac{CA}{2} + \frac{BC}{2} = M_B P_B + M_C M_B = M_C P_B$ . Thus, the perpendicular bisector of  $Q_A P_B$  coincides with the angle bisector of  $\angle M_B M_C M_A$ .

Using similar observations, it may then be concluded that the perpendicular bisectors of  $P_A Q_A, Q_A P_B, P_B Q_B, Q_B P_C, P_C Q_C,$  and  $Q_C P_A$  all concur at the incenter of  $\triangle M_A M_B M_C$ . Thus, the latter must also be the center of the circle containing all six points. As the medial triangle is the image of a homothety on  $\triangle ABC$  with center  $G$  having a scale factor of  $-0.5$ , then the incenter of  $\triangle M_A M_B M_C$  must lie on  $IG$ .  $\square$

5. Find all positive integers  $n$  for which there exists a set of exactly  $n$  distinct positive integers, none of which exceed  $n^2$ , whose reciprocals add up to 1.

Solution. The answer is all  $n \neq 2$ . For  $n = 1$ , the set  $\{1\}$  works. For  $n = 2$ , no set exists, simply because the sum of reciprocals of two distinct integers cannot be equal to 1. For  $n = 3$ , take  $\{2, 3, 6\}$ .

For  $n > 3$ , the identity

$$\frac{1}{k} = \frac{1}{k+r} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots + \frac{1}{(k+r-1)(k+r)}$$

allows us to extend a sum of  $t$  terms to one of exactly  $t + r$  terms. Taking  $k = 3$  and  $r = n - 3$  allows us to turn the sum  $1 = 1/2 + 1/3 + 1/6$  to the  $n$ -term sum

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{n} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n-1)n}.$$

This construction works provided that  $n \neq k(k+1)$  for any  $k$ . Otherwise, we have  $n \geq 6$ , and instead we apply the above to the sum  $1 = 1/2 + 1/3 + 1/10 + 1/15$ , taking  $k = 3$ ,  $r = n - 4$  to yield

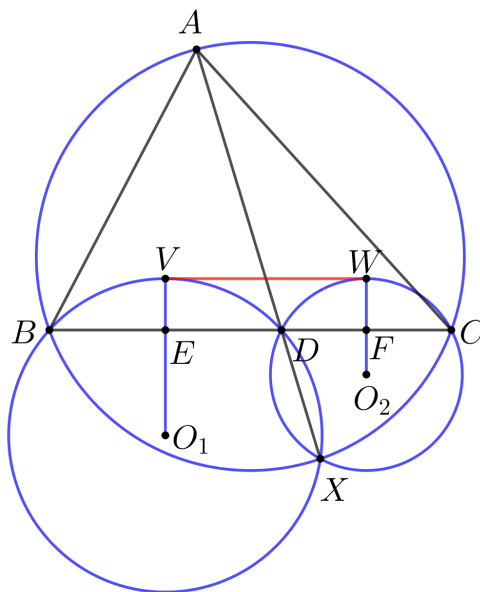
$$1 = \frac{1}{2} + \frac{1}{10} + \frac{1}{15} + \frac{1}{n-1} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n-2)(n-1)}.$$

The above then works, because if  $n = k(k+1)$  for some  $k$ , then  $n - 1 \neq 2, 10, 15$ , and  $n - 1 \neq m(m+1)$  for any  $m$  by parity, since  $n - 1$  is odd and  $m(m+1)$  is always even.

This construction is not unique; there are other similar ones.  $\square$

6. In  $\triangle ABC$ , let  $D$  be the point on side  $BC$  such that  $AB + BD = DC + CA$ . The line  $AD$  intersects the circumcircle of  $\triangle ABC$  again at point  $X \neq A$ . Prove that one of the common tangents of the circumcircles of  $\triangle BD X$  and  $\triangle CD X$  is parallel to  $BC$ .

Solution. Refer to the figure shown below:



Let  $V$  and  $W$  be the midpoints of arcs  $BD$  and  $CD$  respectively. We claim that  $VW$  is the desired common tangent. To prove this, let  $E$  and  $F$  be the orthogonal projections of  $V$  and  $W$  onto  $BC$ . Note that  $E$  and  $F$  are the midpoints of  $BD$  and  $CD$  respectively. Now we claim that  $VE = WF$ . To see this, note that

$$\begin{aligned} VE &= BV \sin \angle VBE \\ &= \frac{BX \sin \angle BXV}{\sin \angle BD X} \sin \frac{\angle BXD}{2} \\ &= \frac{BX \sin \frac{\angle BXA}{2}}{\sin \angle BD X} \sin \frac{C}{2} \\ &= \frac{BX}{\sin \angle BD X} \sin^2 \frac{C}{2}. \end{aligned}$$

Similarly, we can prove that  $WF = \frac{CX}{\sin \angle CDX} \sin^2 \frac{B}{2}$ . Thus, to prove the claim, it suffices to prove that  $\frac{BX}{CX} = \frac{\sin^2 \frac{B}{2}}{\sin^2 \frac{C}{2}}$ . This is because

$$\frac{BX}{CX} = \frac{\sin \angle BAD}{\sin \angle CAD} = \frac{c/BD}{b/CD} = \frac{c(s-c)}{b(s-b)} = \frac{\sin^2 \frac{B}{2}}{\sin^2 \frac{C}{2}}.$$

This proves the first claim.

Next, we claim that  $VW$  is the desired common tangent. Note that from the first claim,  $VWFE$  is a rectangle, since  $\angle VEF$  and  $\angle WFE$  are both right angles. Let  $O_1$  and  $O_2$  be the circumcenters of triangles  $BDX$  and  $CDX$  respectively. Then  $O_1V \perp EF$ , so since  $EF \parallel VW$  we get  $O_1V \perp VW$ . Likewise,  $O_2W \perp VW$  as well, which proves the second claim.

It then follows that  $VW$  is the desired common tangent parallel to  $BC$ , and the required conclusion follows.  $\square$

7. Let  $a, b$ , and  $c$  be positive real numbers such that  $ab + bc + ca = 3$ . Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

Solution. It can be shown that

$$\frac{1}{1+a^4} \geq \frac{2-a^2}{2}.$$

Indeed, simplifying yields

$$\begin{aligned} 2 &\geq (1+a^4)(2-a^2) \\ (a^4+1)(a^2-2)+2 &\geq 0 \\ a^6-2a^4+a^2-2+2 &\geq 0 \\ a^6-2a^4+a^2 &\geq 0 \\ a^2(a^4-2a^2+1) &\geq 0 \\ a^2(a^2-1)^2 &\geq 0 \end{aligned}$$

which is true.

Thus

$$\begin{aligned} \frac{1}{1+a^4} &\geq \frac{2-a^2}{2} \\ \frac{bc}{1+a^4} &\geq \frac{2bc-a^2bc}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \frac{ca}{1+b^4} &\geq \frac{2ca-ab^2c}{2}, \text{ and} \\ \frac{ab}{1+c^4} &\geq \frac{2ab-abc^2}{2}. \end{aligned}$$

Adding these up yields

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq (bc+ca+ab) - \frac{a^2bc+ab^2c+abc^2}{2}.$$

Observe that

$$\begin{aligned} (ab+bc+ca)^2 &\geq 3((ab)(bc) + (bc)(ca) + (ca)(ab)) \\ 9 &\geq 3(a^2bc + ab^2c + abc^2) \\ 3 &\geq a^2bc + ab^2c + abc^2. \end{aligned}$$

Thus,

$$(bc+ca+ab) - \frac{a^2bc+ab^2c+abc^2}{2} \geq 3 - \frac{3}{2} = \frac{3}{2}.$$

Therefore,

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

□

8. The set  $S = \{1, 2, \dots, 2022\}$  is to be partitioned into  $n$  disjoint subsets  $S_1, S_2, \dots, S_n$  such that for each  $i \in \{1, 2, \dots, n\}$ , exactly one of the following statements is true:

- (a) For all  $x, y \in S_i$  with  $x \neq y$ ,  $\gcd(x, y) > 1$ .
- (b) For all  $x, y \in S_i$  with  $x \neq y$ ,  $\gcd(x, y) = 1$ .

Find the smallest value of  $n$  for which this is possible.

Solution. The answer is 15.

Note that there are 14 primes at most  $\sqrt{2022}$ , starting with 2 and ending with 43. Thus, the following partition works for 15 sets. Let  $S_1 = \{2, 4, \dots, 2022\}$ , the multiples of 2 in  $S$ . Let  $S_2 = \{3, 9, 15, \dots, 2019\}$ , the remaining multiples of 3 in  $S$  not in  $S_1$ . Let  $S_3 = \{5, 25, 35, \dots, 2015\}$  the remaining multiples of 5, and so on and so forth, until we get to  $S_{14} = \{43, 1849, 2021\}$ .  $S_{15}$  consists of the remaining elements, i.e. 1 and those numbers with no prime factors at most 43, i.e., the primes greater than 43 but less than 2022:  $S_{15} = \{1, 47, 53, 59, \dots, 2017\}$ . Each of  $S_1, S_2, \dots, S_{14}$  satisfies i., while  $S_{15}$  satisfies ii.

We show now that no partition in 14 subsets is possible. Let a Type 1 subset of  $S$  be a subset  $S_i$  for which i. is true and there exists an integer  $d > 1$  for which  $d$  divides every element of  $S_i$ . Let a Type 2 subset of  $S$  be a subset  $S_i$  for which ii. is true. Finally, let a Type 3 subset of  $S$  be a subset  $S_i$  for which i. is true that is not a Type 1 subset. An example of a Type 3 subset would be a set of the form  $\{pq, qr, pr\}$  where  $p, q, r$  are distinct primes.

Claim: Let  $p_1 = 2, p_2 = 3, p_3 = 5, \dots$  be the sequence of prime numbers, where  $p_k$  is the  $k$ th prime. Every optimal partition of the set  $S(k) := \{1, 2, \dots, p_k^2\}$ , i.e., a partition with the least possible number of subsets, has at least  $k - 1$  Type 1 subsets. In particular, every optimal partition of this set has  $k + 1$  subsets in total. To see how this follows, we look at two cases:

- If every prime  $p \leq p_k$  has a corresponding Type 1 subset containing its multiples, then a similar partitioning to the above works: Take  $S_1$  to  $S_k$  as Type 1 subsets for each prime, and take  $S_{k+1}$  to be everything left over.  $S_{k+1}$  will never be empty, as it has 1 in it. While in fact it is known that, for example, by Bertrand's postulate there is always some prime between  $p_k$  and  $p_k^2$  so  $S_{k+1}$  has at least two elements, there is no need to go this far—if there were no other primes you could just move 2 from  $S_1$  into  $S_{k+1}$ , and if  $k > 1$  then  $S_1$  will still have at least three elements remaining. And if  $k = 1$ , there is no need to worry about this, because  $2 < 3 < 2^2$ .
- On the other hand, if  $p \leq p_k$  has no corresponding Type 1 subset, then  $p$  and  $p^2$  will not be contained in a Type 1 set. Neither can  $p$  nor  $p^2$  be contained in a Type 3 set. If  $\gcd(p, x) > 1$  for all  $x$  in the same set as  $p$ , then  $\gcd(p, x) = p$ , which implies that  $p$  is in a Type 1 set with  $d = p$ . Similarly, if  $\gcd(p^2, x) > 1$  for all  $x$  in the same set as  $p^2$ , then  $p \mid \gcd(p^2, x)$  for all  $x$ , and so  $p^2$  is in a Type 1 set with  $d = p$  as well. Hence  $p$  and  $p^2$  must in fact be in Type 2 sets, and they cannot be in the same Type 2 set (as they share a common factor of  $p > 1$ ); this means that the optimal partition has at least  $k + 1$  subsets in total. A possible equality scenario for example is the sets  $S_1 = \{1, 2, 3, 5, \dots, p_k\}$ ,  $S_2 = \{4, 9, 25, \dots, p_k^2\}$ , and  $S_3$  to  $S_{k+1}$  Type 1 sets taking all remaining multiples of  $2, 3, 5, \dots, p_{k-1}$ . This works, as  $p_k$  and  $p_k^2$  are the only multiples of  $p_k$  in  $S(k)$  with no prime factor other than  $p_k$  and thus cannot be classified into some other Type 1 set.

To prove our claim: We proceed by induction on  $k$ . Trivially, this is true for  $k = 1$ . Suppose now that any optimal partition of the set  $S(k)$  has at least  $k - 1$  Type 1 subsets, and thus at least  $k + 1$  subsets in total. Consider now a partition of the set  $S(k + 1)$ , and suppose that this partition would have at most  $k + 1$  subsets. From the above, there exist at least two primes  $p, q$  with  $p < q \leq p_{k+1}$  for which there are no Type 1 subsets. If  $q < p_{k+1}$  we have a contradiction. Any such partition can be restricted to an optimal partition of  $S(k)$  with  $p < q \leq p_k$  having no corresponding Type 1 subsets. This contradicts our inductive hypothesis. On the other hand, suppose that  $q = p_{k+1}$ . Again restricting to  $S(k)$  gives us an optimal partition of  $S(k)$  with at most  $k - 1$  Type 1 sets; the inductive hypothesis tells us that this partition has in fact exactly  $k - 1$  Type 1 sets and two Type 2 sets from a previous argument establishing the consequence of the claim. However, consider now the element  $pq$ . This cannot belong in any Type 1 set, neither can it belong in the same Type 2 set as  $p$  or  $p^2$ . Thus in addition to the given  $k - 1$  Type 1 sets and 2 Type 2 sets, we need an extra set to contain  $pq$ . Thus our partition of  $S(k + 1)$  in fact has at least  $k - 1 + 2 + 1 = k + 2$  subsets, and not  $k + 1$  subsets as we wanted. The claim is thus proved.

Returning to our original problem, since  $p_{14} = 43 < \sqrt{2022}$ , any partition of  $S$  must restrict to a partition of  $S(14)$ , which we showed must have at least 15 sets. Thus, we can do no better than 15.  $\square$

# 24th PMO Winners



Raphael Dylan T. Dalida  
Philippine Science High School -  
Main Campus

**CHAMPION**



Filbert Ephraim S. Wu  
Victory Christian International School

**1ST RUNNER-UP**



Jerome Austin N. Te  
Jubilee Christian Academy

**2ND RUNNER-UP**

The prizes for the top three contestants in the National Stage are the following:

**CHAMPION** - Medal, trophy, certificate, Php 25,000

**1ST RUNNER-UP** - Medal, trophy, certificate, Php 20,000

**2ND RUNNER-UP** - Medal, trophy, certificate, Php 15,000

Their coaches will receive Php 5,000, Php 3,000, and Php 2,000 respectively. Their schools will also receive a trophy.

Each of the special awardees will be given a medal and certificate. Their respective coaches will also receive a certificate.

**NATIONAL FINALISTS**

# 24th PMO Highlights



Erich Paredes of International School Manila (NCR) taking the PMO Qualifying Stage Exam



Students of Mainit National High School in Surigao del Norte (Region 13)  
(credit: Mr. Val Jorge Stanley Basil)



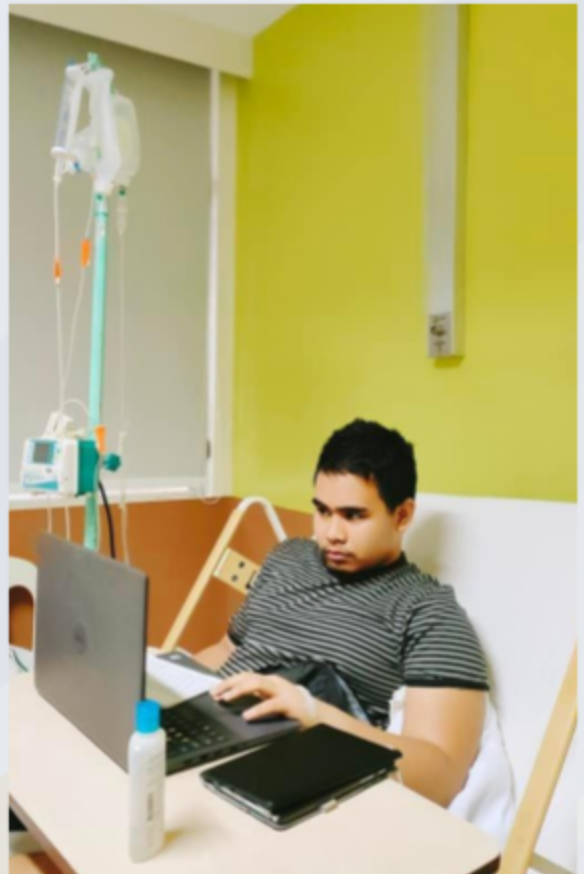
# 24th PMO Highlights



Students of Ozamiz City National High School (Region 10) wearing their PMO shirts (credit: Mr. Nelson Tumala Jr.)



Students of Rawis National High School in Sorsogon City (Region 5) (credit: Mr. Michael Lusuegro)



Josh Robert Obaob of Ateneo de Davao University - Senior High School: in the hospital recovering from his appendicitis operation (Region 11)



Ramon Villota of Dayap National Integrated in Laguna (Region 4A)

# 24th PMO Highlights



Students of Cansilayan Farm School in Negros Occidental (Region 6)  
(credit: Rachel Ricablanca)



Chalaice Cynt Matero of Vivencio P. Casas Sr. Memorial High School (Region 5)  
(credit: Mr. Arjay Lequin)

# 24th PMO Highlights



The PMO National Stage winners with the MSP Board and DOST-SEI representatives during the online awarding ceremony on March 20, 2022, held through Zoom



With the coaches of the National Stage Winners

# 24th PMO Highlights



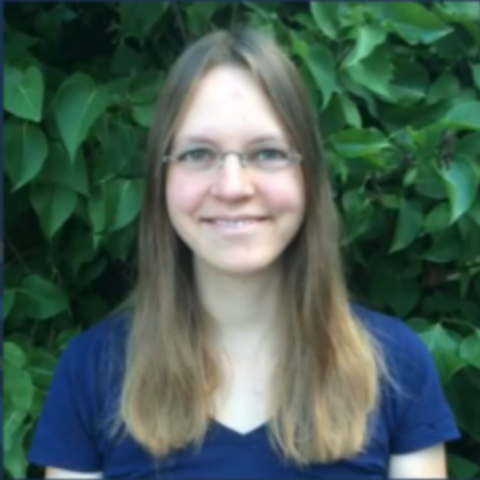
The National Stage finalists with the MSP Board, DOST-SEI representatives, and PMO organizers.



The Regional Coordinators of the PMO

# 24th PMO Highlights

24



**Lisa Sauermann**



International Mathematical Olympiad  
2007 to 2011

Lisa Sauermann announced the Top Female Contestant in the Qualifying Stage. Dr. Sauermann currently teaches at MIT.



She represented Germany in the International Mathematical Olympiad, winning one silver medal and four gold medals. In her last IMO in 2011, she was the only one who got the perfect score.

# 24th PMO Highlights

24

*Nowadays, there's a lot of relevant content online*

*There are more developed training systems in the Philippines.*



Kyle



Albert



Andres

The Philippines' four gold medalists in the IMO announced the Top Junior Contestants in the Qualifying Stage. They are (top to bottom) Farrell Eldrian Wu (IMO 2016 in Hong Kong), Kyle Patrick Dulay (IMO 2016 in Hong Kong), Albert John Patupat (IMO 2018 in Romania), and Andres Rico Gonzales III (IMO 2020 hosted by Russia).



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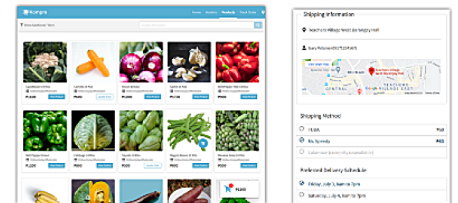
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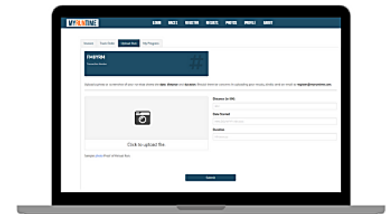
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