# 25th Philippine Mathematical Olympiad Area Stage, 11 February 2023 

PART I. The answer to each item is an integer from 1 to 999 . No solution is needed. Write your answers in the answer sheet. Each correct answer is worth three points.

1. In parallelogram $W X Y Z$, the length of diagonal $W Y$ is 15 , and the perpendicular distances from $W$ to lines $Y Z$ and $X Y$ are 9 and 12, respectively. Find the least possible area of the parallelogram.
2. The product of all real numbers $x$ satisfying $x^{2+\log _{3}(9 x)}=\frac{2187}{x^{2}}$ can be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.
3. Let $x$ and $y$ be integers satisfying $x^{2}+30 x+25=y^{4}$. What is the largest possible value of $x+y$ ?
4. A game is played on the number line. Initially, there is a token placed at the number 0 . In each move, the player can move the token from its current position $x$, to either $x+2023$ or $x-59$. The goal of the game is to move the token to either 1 or -1 . What is the minimum number of moves required to achieve this goal?
5. Let $x, y, z$ be three real numbers such that

- $y, x, z$ form a harmonic sequence; and
- $3 x y, 5 y z, 7 z x$ form a geometric sequence.

The numerical value of $\frac{y}{z}+\frac{z}{y}$ can be expressed in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. What is $p+q$ ?
6. There are three novel series Peter wishes to read. Each consists of 4 volumes that must be read in order, but not necessarily one after the other. Let $N$ be the number of ways Peter can finish reading all the volumes. Find the sum of the digits of $N$. (Assume that he must finish a volume before reading a new one.)
7. Suppose that $P(x)$ and $Q(x)$ are both quadratic polynomials with leading coefficient 1 such that $P(P(x)-x)=(Q(x))^{2}$ for all real numbers $x$ and $P(2)=0$. Find the sum of all possible values of $P(10)$.
8. Let $S$ be the sum of all positive integers less than $10^{6}$ which can be expressed as $m!+n!$, where $m$ and $n$ are nonnegative integers. Determine the last three digits of $S$.
9. How many 9 -term sequences $a_{1}, \ldots, a_{9}$ of nonnegative integers are there such that

- $0 \leq a_{i}<i$ for all $i=1, \ldots, 9$; and
- there are no ordered triples $(i, j, k)$ with $1 \leq i<j<k \leq 9$, such that $a_{i}, a_{j}, a_{k}$ are all distinct?

10. Suppose $P Q R S$ is a convex quadrilateral with $\angle S P Q=\angle P Q R=120^{\circ}, S P-Q R=36$, $R S=84$, and $Q R$ is a positive even integer. Let $T$ be the intersection of lines $S P$ and $Q R$. What is the largest possible perimeter of $\triangle P Q T$ ?
11. In square $A B C D, P$ lies on the ray $A D$ past $D$ and lines $P C$ and $A B$ meet at $Q$. Point $X$ is the foot of the perpendicular from $B$ to $D Q$, and the circumcircle of triangle $A P X$ meets line $A B$ again at $Y$. Suppose that $D P=\frac{16}{3}$ and $B Q=27$. The length of $B Y$ can be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.
12. Seven people are seated together around a circular table. Each one will toss a fair coin. If the coin shows a head, then the person will stand. Otherwise, the person will remain seated. The probability that after all of the tosses, no two adjacent people are both standing, can be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. What is $p+q$ ?
13. Let $a, b, c$ be real numbers with $1<a<b<c$ that satisfy the equations

$$
\begin{gathered}
\log _{a} b+\log _{b} c+\log _{c} a=6.5 \\
\log _{b} a+\log _{c} b+\log _{a} c=5 .
\end{gathered}
$$

Then $\max \left\{\log _{a} b, \log _{b} c, \log _{c} a\right\}$ can be written in the form $\sqrt{x}+\sqrt{y}$, where $x$ and $y$ are positive integers. What is $x+y$ ?
14. In triangle $A B C$ with orthocenter $H, A B=13, B C=21$ and $C A=20$. The perpendicular bisector of $C H$ meets $B C$ at $P$ and lines $P H$ and $A B$ meet at $Q$. The line through $Q$ perpendicular to $P Q$ meets $A H$ at $X$. The length of $A X$ can be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.
15. Bryce has 7 blue socks and 7 red socks mixed in a drawer. He plays a game with Sean. Blindfolded, Bryce takes two socks from the drawer. Sean looks at the socks, and if they have the same color, Sean gives Bryce 1 point. Bryce keeps drawing socks until the drawer is empty, at which time the game ends. The probability that Bryce's score is at most 2 can be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.
16. Three circles $\Gamma_{1}, \Gamma_{2}, \Gamma_{3}$ are pairwise externally tangent, with $\Gamma_{1}$, the smallest, having radius 1 , and $\Gamma_{3}$, the largest, having radius 25 . Let $A$ be the point of tangency of $\Gamma_{1}$ and $\Gamma_{2}, B$ be the point of tangency of $\Gamma_{2}$ and $\Gamma_{3}$, and $C$ be the point of tangency of $\Gamma_{1}$ and $\Gamma_{3}$. Suppose now that triangle $A B C$ has circumradius 1 as well. The radius of $\Gamma_{2}$ can then be written in the form $p / q$, where $p$ and $q$ are relatively prime positive integers. Find the value of the product $p q$.
17. For each positive integer $n$, define the function $f_{n}(x)=|n-x|$. How many real solutions are there to

$$
\left(f_{1} \circ f_{2} \circ \cdots \circ f_{24} \circ f_{25}\right)(x)=0 ?
$$

18. Suppose that $p$ is a prime number which divides infinitely many numbers of the form $10^{n!}+2023$ where $n$ is a positive integer. What is the sum of all possible values of $p$ ?
19. Let $p=p_{1} p_{2} \ldots p_{6}$ be a permutation of the integers from 1 to 6 . For any such permutation $p$, we count how many integers there are which have nothing bigger on its left. We let $f(p)$ be the number of these integers for the permutation $p$. For example, $f(612345)=1$ because only 6 has nothing to its left which is bigger. On the other hand, $f(135462)=4$ because only $1,3,5$, and 6 satisfy the condition.
Let $S$ be the sum of $f(p)$ over all the 6 ! different permutations. Find the sum of the digits of $S$.
20. Determine the sum of all positive integers $n$ for which $2[\tau(n)]^{2}=3 n$, where $\tau(n)$ is the number of positive divisors of $n$.

PART II. Write your solutions to each problem in the solution sheets. Each complete and correct solution is worth ten points.

1. For a set of real numbers $A$, let $A-A$ be the set of distinct pairwise differences of $A$; that is,

$$
A-A:=\{a-b: a, b \in A\}
$$

If $|A-A|=25$, find the sum of all possible values of $|A|$.
2. Let $A B C$ be an acute scalene triangle with orthocenter $H$. Let $M$ be the midpoint of $B C$, and suppose that the line through $H$ perpendicular to $A M$ intersects $A B$ and $A C$ at points $E$ and $F$ respectively. Denote by $O$ the circumcenter of triangle $A E F$, and $D$ the foot of the perpendicular from $H$ to $A M$. Prove that the line $A O$ intersects the perpendicular from $D$ to $B C$ at a point on the circumcircle of triangle $A B C$.

## Answers to the 25th PMO Area Stage

Part I. (3 points each)

1. 108
2. $65(48 / 17)$
3. 730
4. $157(29 / 128)$
5. 43
6. 16
7. 247
8. 173 (119/54)
9. 59
(34/25)
10. 613
(184/429)
11. 18
(34650)
12. 64
13. 130 (4091130)
14. 34
15. 503
16. 18
(1764)
17. 174
18. 96

## Part II. (10 points each)

1. For a set of real numbers $A$, let $A-A$ be the set of distinct pairwise differences of $A$; that is,

$$
A-A:=\{a-b: a, b \in A\}
$$

If $|A-A|=25$, find the sum of all possible values of $|A|$.
Answer: 76
Solution. First, we show that $6 \leq|A| \leq 13$. To do this, we show that if $|A|=n$, then $2 n-1 \leq|A-A| \leq n(n-1)+1$. The upper bound is easily attained; the number of distinct differences is bounded above by the number of ordered pairs $(a, b)$ of distinct elements of $A$ plus 1 , to account for the remaining pairs $(a, a)$ which give a difference of 0 ; this is exactly $n(n-1)+1$. For the lower bound, suppose that $A=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ with $a_{1}<a_{2}<\cdots<a_{n}$. Then the $n-1$ differences $a_{n}-a_{1}, a_{n}-a_{2}, \ldots, a_{n}-a_{n-1}$ are all positive and distinct and appear in $A-A$; their additive inverses are all negative and distinct and also appear in $A-A$. Also, $0 \in A-A$ and 0 is not among those differences previously listed. This gives us at least $2(n-1)+1=2 n-1$ differences in $A-A$. These bounds show us that if $|A-A|=25$, we must have $6 \leq|A| \leq 13$ as desired.
We then show that for any $n$ with $6 \leq n \leq 13$, there exists a set $A$ such that $|A|=n$ and $|A-A|=25$. Consider now the sets $B=\{1,2,4,8,12,13\}$ and $C=\{1,2, \ldots, 13\}$. Then we have $|B|=6,|C|=13, B \subseteq C$, and $B-B=C-C=\{-12,-11, \ldots, 0, \ldots, 11,12\}$. Now, for any set $A$ with $B \subseteq A \subseteq C$, we have $B-B \subseteq A-A \subseteq C-C$, and so we must have $A-A=B-B=C-C$. For any $n$ with $6 \leq n \leq 13$, it is always possible to choose $A$ such that $|A|=n$ and $B \subseteq A \subseteq C$, and this will give us the desired $A$.
Thus, the sum of all possible values of $|A|$ is $6+7+\cdots+13=76$.
2. Let $A B C$ be an acute scalene triangle with orthocenter $H$. Let $M$ be the midpoint of $B C$, and suppose that the line through $H$ perpendicular to $A M$ intersects $A B$ and $A C$ at points $E$ and $F$ respectively. Denote by $O$ the circumcenter of triangle $A E F$, and $D$ the foot of the perpendicular from $H$ to $A M$. Prove that the line $A O$ intersects the perpendicular from $D$ to $B C$ at a point on the circumcircle of triangle $A B C$.

Solution. WLOG assume $A B<A C$. Let $A M$ intersect the circumcircle of $A B C$ again at $Y \neq A$. We first need to prove a claim.


Claim: Quadrilateral BHDC is cyclic.
Proof of Claim: Consider the reflection with respect to $M$. This maps $B$ and $C$ to each other. It is fairly well-known that this also maps the orthocenter $H$ to the antipode of $A$ with respect to $(A B C)$, which we will denote by $H^{\prime}$. Thus, $(B H C)$ and $\left(C H^{\prime} B\right)$ are mapped to each other. Since $\angle H D M$ and $\angle H^{\prime} Y M$ are both right angles with $M$ the midpoint of $H H^{\prime}$, then $\triangle H D M$ and $\triangle H^{\prime} Y M$ are congruent right triangles. It follows that $D$ is the reflection of $Y$ with respect to $M$. Since $Y \in\left(C H^{\prime} B\right)$, then $D \in(B H C)$, and the claim follows.

Now let $A O$ intersect $(A B C)$ and $(A E F)$ again at $X$ and $Z$, respectively. Then

$$
\angle B A X=\angle E A Z=90^{\circ}-\angle A Z E=90^{\circ}-\angle A F E=\angle D A F=\angle Y A C
$$

and so $X Y \| B C$.
Consider reflection with respect to $B C$. This maps $(B H C)$ to $(A B C)$. Let $X^{\prime}$ be the image of $D$, so $X^{\prime} \in(A B C)$ and $D X^{\prime} \perp B C$. Let $N$ be the midpoint of $D X^{\prime}$, which is on $B C$. Then note that $X^{\prime} Y \| N M$. Since $X, X^{\prime} \in(A B C)$, and both $X Y$ and $X^{\prime} Y$ are parallel to $B C$, then $X=X^{\prime}$. The desired conclusion follows.

