$25^{\text {th }}$ Philippine Mathematical Olympiad
National Stage (Day 1)
18 March 2023

Time: 4.5 hours
Each item is worth 7 points.

1. Find all ordered pairs $(a, b)$ of positive integers such that $a^{2}+b^{2}+25=15 a b$ and $a^{2}+a b+b^{2}$ is prime.
2. Find all primes $p$ such that $\frac{2^{p+1}-4}{p}$ is a perfect square.
3. In $\triangle A B C, A B>A C$. Point $P$ is on line $B C$ such that $A P$ is tangent to its circumcircle. Let $M$ be the midpoint of $A B$, and suppose the circumcircle of $\triangle P M A$ meets line $A C$ again at $N$. Point $Q$ is the reflection of $P$ with respect to the midpoint of segment $B C$. The line through $B$ parallel to $Q N$ meets $P N$ at $D$, and the line through $P$ parallel to $D M$ meets the circumcircle of $\triangle P M B$ again at $E$. Show that the lines $P M$, $B E$, and $A C$ are concurrent.
4. In chess, a knight placed on a chess board can move by jumping to an adjacent square in one direction (up, down, left, or right) then jumping to the next two squares in a perpendicular direction. We then say that a square in a chess board can be attacked by a knight if the knight can end up on that square after such a move. Thus, depending on where a knight is placed, it can attack as many as eight squares, or maybe even less.
In a $10 \times 10$ chess board, what is the maximum number of knights that can be placed such that each square on the board can be attacked by at most one knight?
$25^{\text {th }}$ Philippine Mathematical Olympiad
National Stage (Day 2)
19 March 2023

Time: 4.5 hours
Each item is worth 7 points.
5. Silverio is very happy for the 25 th year of the PMO. In his jubilation, he ends up writing a finite sequence of As and Gs on a nearby blackboard. He then performs the following operation: if he finds at least one occurrence of the string "AG", he chooses one at random and replaces it with "GAAA". He performs this operation repeatedly until there is no more "AG" string on the blackboard. Show that for any initial sequence of As and Gs, Silverio will eventually be unable to continue doing the operation.
6. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
f(2 f(x))=f(x-f(y))+f(x)+y
$$

for all $x, y \in \mathbb{R}$.
7. A set of positive integers is said to be pilak if it can be partitioned into 2 disjoint subsets $F$ and $T$, each with at least 2 elements, such that the elements of $F$ are consecutive Fibonacci numbers, and the elements of $T$ are consecutive triangular numbers. Find all positive integers $n$ such that the set containing all the positive divisors of $n$ except $n$ itself is pilak.
8. Let $\mathcal{S}$ be the set of all points in the plane. Find all functions $f: \mathcal{S} \rightarrow \mathbb{R}$ such that for all nondegenerate triangles $A B C$ with orthocenter $H$, if $f(A) \leq f(B) \leq f(C)$, then

$$
f(A)+f(C)=f(B)+f(H) .
$$

## Solutions to the 25th Philippine Mathematical Olympiad

1. Find all ordered pairs $(a, b)$ of positive integers such that $a^{2}+b^{2}+25=15 a b$ and $a^{2}+a b+b^{2}$ is prime.

Solution. The only solutions are $(1,2)$ and $(2,1)$.
First, note that $a \neq b$; otherwise we have $13 a^{2}=25$ which is impossible. We can assume without loss of generality that $1 \leq a<b$; we then take $(b, a)$ as well.
If $a=1$, we get $b^{2}+26=15 b$ which yields $b=2$ or $b=13$. For $b=2$ we get $a^{2}+a b+b^{2}=7$ which is indeed prime; for $b=13$ however we have $a^{2}+a b+b^{2}=183=3 \cdot 61$ which is composite.
If $a=2$, we get $b^{2}+29=30 b$, which yields $b=29$; we have $a^{2}+a b+b^{2}=903=3 \cdot 7 \cdot 43$ which is composite as well.
Now, suppose $a>2$. Note that $17\left(a^{2}+a b+b^{2}\right)=16 a^{2}+32 a b+16 b^{2}-25=(4 a+4 b-$ 5) $(4 a+4 b+5)$. With the condition that $a \geq 3, b \geq 4$, we have $4 a+4 b+5>4 a+4 b-5 \geq$ $23>17$, that is, $17\left(a^{2}+a b+b^{2}\right)$ is the product of two positive integers greater than 17. Thus, $a^{2}+a b+b^{2}$ is the product of two positive integers greater than 1 , and so is composite.
This tells us that indeed $(1,2)$ and $(2,1)$ are the only solutions.
2. Find all primes $p$ such that $\frac{2^{p+1}-4}{p}$ is a perfect square.

Solution. Note that the given equation can be rewritten as follows.

$$
\frac{2^{p+1}-4}{p}=4 n^{2} \quad \Longrightarrow \quad \frac{2^{p-1}-1}{p}=n^{2}
$$

Note that if $p=2$, the only even prime number, then $\frac{2^{p-1}-1}{p}=\frac{1}{2}$, which is not a square of an integer. Thus, $p$ must be odd, and so, $p-1$ is even, which means we can factor the numerator because it is a difference of two squares. Hence,

$$
\frac{2^{p-1}-1}{p}=\frac{\left(2^{\frac{p-1}{2}}-1\right)\left(2^{\frac{p-1}{2}}+1\right)}{p} .
$$

We note that the two factors in the numerator differ by 2 and each factor is odd. Thus,

$$
\operatorname{gcd}\left(2^{\frac{p-1}{2}}-1,2^{\frac{p-1}{2}}+1\right)=1
$$

This implies that

$$
\operatorname{gcd}\left(2^{\frac{p-1}{2}}-1, \frac{2^{\frac{p-1}{2}}+1}{p}\right)=\operatorname{gcd}\left(2^{\frac{p-1}{2}}+1, \frac{2^{\frac{p-1}{2}}-1}{p}\right)=1 .
$$

Since we also have $\frac{\left(2^{\frac{p-1}{2}}+1\right)}{p}\left(2^{\frac{p-1}{2}}-1\right)=n^{2}$, then both $\frac{\left(2^{\frac{p-1}{2}}+1\right)}{p}$ and $\left(2^{\frac{p-1}{2}}-1\right)$ or both $\frac{\left(2^{\frac{p-1}{2}}-1\right)}{p}$ and $\left(2^{\frac{p-1}{2}}+1\right)$ must be perfect squares.
Case 1: Let $2^{\frac{p-1}{2}}-1=x^{2}$, where $x=2 q+1$ with $q \in \mathbb{Z}$. Then

$$
2^{\frac{p-1}{2}}-1=(2 q+1)^{2}=4 q^{2}+4 q+1=4 q(q+1)+1 \Longrightarrow 2^{\frac{p-1}{2}}-2=4 q(q+1)
$$

If $\frac{p-1}{2} \geq 2$, then $2^{\frac{p-1}{2}}$ will be divisible by 4 , but $2^{\frac{p-1}{2}}-2$ will be $2(\bmod 4)$, which is a contradiction. Thus, $\frac{p-1}{2}<2$, and so $2<p<5$. Hence, $p=3$ is only possible case. If $p=3$, then $\frac{2^{p-1}-1}{p}=1$, which is a perfect square.

Case 2: Let $2^{\frac{p-1}{2}}-1=y^{2}$, where $y=2 r+1$ with $r \in \mathbb{Z}$. Then

$$
2^{\frac{p-1}{2}}+1=(2 r+1)^{2}=4 r^{2}+4 r+1=4 r(r+1)+1 \Longrightarrow 2^{\frac{p+1}{2}}=4 r(r+1) .
$$

If $r>1$, then $r$ or $r+1$ is an odd number, which means $4 r(r+1)$ has an odd divisor, but $2^{\frac{p-1}{2}}$ is even for all $p \geq 3$. Thus, $r=1$ is the only possible case. Hence, we have

$$
2^{\frac{p-1}{2}}=4(1)(2)=8 \Longrightarrow \frac{p-1}{2}=3 \Longrightarrow p=7 .
$$

If $p=7$, then $\frac{2^{p-1}-1}{p}=9$, which is a perfect square.
Since we have exhausted all possible cases, then $p=3,7$ are the only possible values of $p$.
3. In $\triangle A B C, A B>A C$. Point $P$ is on line $B C$ such that $A P$ is tangent to its circumcircle. Let $M$ be the midpoint of $A B$, and suppose the circumcircle of $\triangle P M A$ meets line $A C$ again at $N$. Point $Q$ is the reflection of $P$ with respect to the midpoint of segment $B C$. The line through $B$ parallel to $Q N$ meets $P N$ at $D$, and the line through $P$ parallel to $D M$ meets the circumcircle of $\triangle P M B$ again at $E$. Show that the lines $P M, B E$, and $A C$ are concurrent.

## Solution.



Let lines $B E$ and $A C$ meet at $R$. It suffices to show that the points $P, M$ and $R$ are collinear. Since $A P$ is tangent to the circumcircle of $A B C$ and $P N A M$ is cyclic, we have $\angle P N M=\angle P A B=\angle A C B$ and $\angle B A C=\angle M P N$. Thus, triangles $M P N$ and $B A C$ are similar with $\frac{M P}{M N}=\frac{A B}{B C}$. Also, since $D M$ and $P E$ are parallel and $D P E M$ is cyclic,
we get $\angle P M D=\angle M P E=\angle A B R$, so that triangles $P M D$ and $A B R$ are similar with $\frac{P D}{P M}=\frac{A R}{A B}$.
Now, observe that $\angle P E R=\angle P E B=180^{\circ}-\angle P M B=180^{\circ}-\angle P N R$, so $P E R N$ is cyclic and $\angle C R B=\angle D P E=\angle N D M$. We see that triangles $C R B$ and $N D M$ are similar with $\frac{C B}{C R}=\frac{M N}{N D}$. With $Q C=P B$, we have $Q B=P C$ and Thales' theorem gives $\frac{P D}{D N}=\frac{P B}{B Q}=\frac{P B}{P C}$. Thus, applying Menelaus' theorem on triangle $P A C$, we obtain

$$
\frac{C R}{A R} \cdot \frac{A M}{M B} \cdot \frac{B P}{P C}=\frac{C B \cdot N D}{M N} \cdot \frac{P M}{P D \cdot A B} \cdot \frac{P D}{D N}=\frac{P M}{M N} \cdot \frac{B C}{A B}=1,
$$

implying that $P, M$ and $R$ are collinear as desired.
4. In chess, a knight placed on a chess board can move by jumping to an adjacent square in one direction (up, down, left, or right) then jumping to the next two squares in a perpendicular direction. We then say that a square in a chess board can be attacked by a knight if the knight can end up on that square after such a move. Thus, depending on where a knight is placed, it can attack as many as eight squares, or maybe even less.

In a $10 \times 10$ chess board, what is the maximum number of knights that can be placed such that each square on the board can be attacked by at most one knight?

Solution. The answer is 16 . 16 knights can be placed as follows, where a blue square represents a square with a knight:


We will now show it is not possible to place more than 16 knights. First we color the grid black and white in checkerboard fashion:


Note that each knight on a black square can only attack white squares, and each knight on a white square can only attack black squares. If there are more than 16 knights on the grid, there must be at least 9 knights on white squares or at least 9 knights on black squares. Without loss of generality, let there be at least 9 knights on black squares. Let us split the $10 \times 10$ grid into four $5 \times 5$ grids. Note that two of those four have more black squares than white squares (we call this $A$ ), and the other two have more white squares than black squares (we call this $B$ ).


We see that in grids of type $A$, we can only place at most 3 knights on black squares (center and two opposite corners), and in grids of type $B$, we can only place at most 2 knights on black squares ( 2 squares horizontal and 2 squares vertical from one another). Since there are at least 9 knights on four $5 \times 5$ grids, at least one of those grids has at least 3 knights (which must be type $A$ ). Assume that the $A$ grid on the bottom left corner has 3 knights on black squares. Then we will have two cases (red squares indicate black squares which we cannot place a knight on):


Note that in both cases, the top left and bottom right $B$ grids can only each have at most 1 knight on a black square, while the top right $A$ grid can have at most 3 knights on black squares. This means we can have at most $3+1+1+3=8$ knights on black squares in the $10 \times 10$ grid, contradiction. Therefore, 16 is the maximum.
5. Silverio is very happy for the 25th year of the PMO. In his jubilation, he ends up writing a finite sequence of As and Gs on a nearby blackboard. He then performs the following operation: if he finds at least one occurrence of the string "AG", he chooses one at random and replaces it with "GAAA". He performs this operation repeatedly until there is no more "AG" string on the blackboard. Show that for any initial sequence of As and Gs, Silverio will eventually be unable to continue doing the operation.

Solution: We assign the weight $4^{k}$ to each B in the sequence, where $k$ is the number of A's to the right of this B. In each operation, if $4^{k}$ is the weight of the B in the "BA" being replaced, then each of the three B's in "ABBB" have a weight of $4^{k-1}$. So the sum of the weights decrease by $4^{k}-3 \cdot 4^{k-1}=4^{k-1}$ in each operation. Since the sum of weights in the initial sequence is finite, and the sum of the weights of all B's must be a nonnegative integer, Steve can only perform a finite number of operations.
6. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
f(2 f(x))=f(x-f(y))+f(x)+y
$$

for all $x, y \in \mathbb{R}$.
Solution. Let $P(x, y)$ be the problem statement. Note that if $f(a)=f(b)$ where $a, b \in \mathbb{R}$, then $P(x, a)$ and $P(x, b)$ imply $a=b$. Thus $f$ is injective. Then $P(x,-f(x))$ implies $f(2 f(x))=f(x-f(-f(x)))$, so by injectivity we obtain $f(-f(x))=x-2 f(x)$.
Now $P(x, 0)$ gives $f(2 f(x))=f(x-f(0))+f(x)$. Combining this with $P(x, y)$, we obtain $f(x-f(0))=f(x-f(y))+y$. Setting $x=0$ here yields $f(-f(0))=f(-f(y))+y$, and so $f(-f(0))=2 y-f(y)$. In particular, this implies $f(y)=y+c$ for some constant $c$. Substituting this into $P(x, y)$, we obtain

$$
2 x+3 c=x-y+x+y+c
$$

which implies that $c=0$. It is easy to see that $f(x)=x$ works and is therefore the only solution.
7. A set of positive integers is said to be pilak if it can be partitioned into 2 disjoint subsets $F$ and $T$, each with at least 2 elements, such that the elements of $F$ are consecutive Fibonacci numbers, and the elements of $T$ are consecutive triangular numbers. Find all positive integers $n$ such that the set containing all the positive divisors of $n$ except $n$ itself is pilak.

## Solution.

The only positive integer $n$ that satisfies this property is 30 .
In this case, $D=\{1,2,3,5,6,10,15\}$, and we can partition $D$ into $F=\{1,2,3,5\}$ and $T=\{6,10,15\}$. We will show that there are no other $n$.

Claim 1: $1 \in F$.
Proof: Obviously, $1 \in D$. Assume that $1 \in T$. Since $T$ has at least 2 elements, $3 \in T$. If $6 \in T$, then $2 \in D$. Note that 2 is not a triangular number. If $2 \in F$, then since $F$ has at least 2 elements, at least one of 1 or 3 must be in $F$. But both 1 and 3 are in $F$, contradiction. Thus, $6 \notin T$, and $T=\{1,3\}$.
Now, if we have two consecutive Fibonacci numbers $F_{k}$ and $F_{k+1}$ in $F$, at least one of them is not divisible by 3 , so at least one of $3 F_{k}$ or $3 F_{k+1}$ divides $n$. It is well known that two consecutive Fibonacci numbers are relatively prime, so $F_{k+1} \mid 3 F_{k}$ and $F_{k} \mid 3 F_{k+1}$ are both impossible. Thus, $n \neq 3 F_{k}, 3 F_{k+1}$, implying that least one of $3 F_{k}$ or $3 F_{k+1}$ is in $F$. However, for all integers $i>1, F_{i+3}=F_{i+2}+F_{i+1}=2 F_{i+1}+F_{i}>3 F_{i}>2 F_{i}+F_{i-1}=$ $F_{i}+F_{i+1}=F_{i+2}$, so $3 F_{i}$ cannot be a Fibonacci number, and none of $3 F_{k}$ or $3 F_{k+1}$ is in
$F$, contradiction. Therefore, $1 \in F$.
Claim 2: $3 \in F$.
If $1 \in F$, then since $F$ has at least two elements, $2 \in F$. If $3 \notin F$, then $F=\{1,2\}$. Let $t$ denote the smallest number in $T$. If $t$ is composite, since 4 is not triangular, there exists a proper divisor of $t$ greater than 2 . Since this is a proper divisor of $n$, smaller than $t$, and greater than 2 , this number cannot be in either $F$ or $T$, contradiction. Thus, $t$ is prime. However, 3 is the only triangular prime number, so $t=3$ and $3 \in T$. Since $1 \notin T$ and $T$ has at least two elements, $6 \in T$. If $10 \in T$, then $5 \in D$. Since 5 is not a triangular number, and $5 \notin F$, we have a contradiction. Thus, $10 \notin T$, and $T=\{3,6\}$. This implies that $D=\{1,2,3,6\}$, and that $n$ has exactly 5 divisors, making $n$ a perfect square. $n$ is even, thus it must be divisible by 4 . However, $4 \notin D$, contradiction. Therefore, $3 \in F$.

Since $1,3 \in F, 2 \in F$ and $6 \mid n$. Note that $n$ must have at least 5 positive integer divisors, and it is easy to check that among $1,2, \ldots, 15$, only 12 has at least 5 positive integer divisors. Thus, $n>6$ and $6 \in D$. Since 6 is not a Fibonacci number, $6 \in T$. Since $3 \notin T$ and $T$ has at least two elements, $10 \in T$. This also implies that $5 \in D$. Since 5 is not a triangular number, $5 \in F$. So $15 \mid n$. Since $5 \nmid 12, n \neq 12$, so $n>15$, making $15 \in D$. Since 15 is not a Fibonacci number, $15 \in T$. If $21 \in T$, then $7 \in D$. However, 7 is neither a Fibonacci number nor a triangular number, contradiction. Thus, $T=\{6,10,15\}$. If $8 \in F$, then $4 \in D$. However, 4 is neither a Fibonacci number nor a triangular number, contradiction. Thus, $F=\{1,2,3,5\}$.

Since $D=\{1,2,3,5,6,10,15\}, n=30$ is the only possible solution.
8. Let $\mathcal{S}$ be the set of all points in the plane. Find all functions $f: \mathcal{S} \rightarrow \mathbb{R}$ such that for all nondegenerate triangles $A B C$ with orthocenter $H$, if $f(A) \leq f(B) \leq f(C)$, then

$$
f(A)+f(C)=f(B)+f(H) .
$$

Solution. Let $P(A, B, C)$ be the problem assertion. First consider a non-right triangle $A B C$ with orthocenter $H$. Note that in the set $\{A, B, C, H\}$, the last point is the orthocenter of the other three. Thus, we can assume WLOG $f(A) \leq f(B) \leq f(C) \leq$ $f(H)$.
By considering $P(A, B, C)$, this implies $f(A)+f(C)=f(B)+f(H)$. Since $f(A) \leq f(B)$ and $f(C) \leq f(H)$, this implies equality must hold in both inequalities, and so $f(A)=$ $f(B)$ and $f(C)=f(H)$.
Denote by $\Omega_{B C}$ the circle with diameter $B C$. We prove the following claim.
Claim: If $f(B) \neq f(C)$, then for all $D \in \Omega_{B C}$, we have $2 f(D)=f(B)+f(C)$. In particular, $f$ is constant on $\Omega_{B C}$.
Proof: We split into cases. If $f(D) \notin[f(B), f(C)]$, then $P(B, C, D)$ implies

$$
f(D)+f(C)=f(B)+f(D),
$$

which implies $f(B)=f(C)$, contradiction. Thus, $f(D) \in[f(B), f(C)]$, from which $P(B, C, D)$ implies

$$
f(B)+f(C)=2 f(D)
$$

The claim then follows.
Now we claim that $f(B)=f(C)$. Assume for the sake of contradiction that this was not the case. Consider $\Omega_{A C}$, and let $D=A H \cap B C \in \Omega_{A C}$. Let $\Omega_{B D^{\prime}}$ intersect $\Omega_{A C}$ at a second point $D^{\prime \prime} \neq D^{\prime}$. Then since $f(A) \neq f(C)$, from the claim we get $2 f\left(D^{\prime}\right)=f(A)+f(C)$.
Now we have two cases. In the first case, suppose $f(B)=f\left(D^{\prime}\right)$. Then $2 f(B)=f(A)+$ $f(C)$, implying $f(B)=f(C)$, contradiction. In the second case, we have $f(B)+f\left(D^{\prime}\right)=$ $2 f\left(D^{\prime \prime}\right)=f(C)+f(A)$. This implies $f(C)=f\left(D^{\prime}\right)$, and so

$$
2 f(C)=2 f\left(D^{\prime}\right)=f(C)+f(A)
$$

implying $f(C)=f(A)=f(B)$, another contradiction. Thus our assumption was wrong, and $f(B)=f(C)$. This implies $f(A)=f(B)=f(C)=f(H)$ for any nonnedegerate non-right triangle $A B C$, and so $f$ is constant everywhere (by considering two segments whose diameter circles do not intersect). It is clear that these solutions work.

