25th Philippine Mathematical Olympiad

Qualifying Stage, 14 January 2023

PART I. Choose the best answer. Each correct answer is worth two points.

1. How many four-digit numbers contain the digit 5 or 7 (or both)?

(a) 5416 (b) 5672 (c) 5904 (d) 6416

2. Let $O(0,0)$ and $A(0,1)$. Suppose a point B is chosen (uniformly) at random on the circle $x^2+y^2=1$. What is the probability that OAB is a triangle whose area is at least $\frac{1}{4}$?

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

3. Suppose $a_1 < a_2 < \cdots < a_{25}$ are positive integers such that the average of $a_1, a_2, \ldots, a_{24}$ is one-half the average of $a_1, a_2, \ldots, a_{25}$. What is the minimum possible value of a_{25}?

(a) 26 (b) 275 (c) 299 (d) 325

4. Suppose that a real-valued function $f(x)$ has domain $(-1,1)$. What is the domain of the function $f\left(\dfrac{3-x}{3+x}\right)$?

(a) $(0,+\infty)$ (b) $(-3,3)$ (c) $(-\infty,-3)$ (d) $(-\infty,-3)\cup(3,+\infty)$

5. Aby chooses a positive divisor a of 120 (uniformly) at random. Brian then chooses a positive divisor b of a (uniformly) at random. What is the probability that b is odd?

(a) $\frac{2}{5}$ (b) $\frac{13}{24}$ (c) $\frac{15}{32}$ (d) $\frac{25}{48}$

6. Find the sum of the squares of all integers n for which $\sqrt{\dfrac{4n+25}{n-20}}$ is an integer.

(a) 466 (b) 475 (c) 2306 (d) 2531

7. Let $k>1$. The graphs of the functions $f(x)=\log\left(\sqrt{x^2+k^3}+x\right)$ and $g(x)=2\log\left(\sqrt{x^2+k^3}-x\right)$ have a unique point of intersection (a,b). Find $2a$.

(a) $\sqrt{k^3-k+1}$ (b) $k^{3/2}-k^{1/2}+1$ (c) k^2+k+1 (d) k^2-k

8. The sides of a convex quadrilateral have lengths 12 cm, 12 cm, 16 cm, and 16 cm, and they are arranged so that there are no pairs of parallel sides. If one of the diagonals is 20 cm long, and the length of the other diagonal is a rational number, what is the length of the other diagonal?

(a) $\frac{48}{5}$ cm (b) $\frac{84}{5}$ cm (c) $\frac{96}{5}$ cm (d) $\frac{108}{5}$ cm

9. How many numbers from 1 to 10^4 can be expressed both as a sum of five consecutive positive integers and as a sum of seven consecutive positive integers, but not as a sum of three consecutive positive integers?

(a) 142 (b) 190 (c) 285 (d) 2096

10. For positive real numbers a and b, the minimum value of

$$\left(18a + \frac{1}{3b}\right)\left(3b + \frac{1}{8a}\right)$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. The value of $m + n$ is

(a) 29 (b) 27 (c) 13 (d) 7

11. In $\triangle ABC$, let D be a point on BC such that $BD : BC = 1 : 3$. Given that $AB = 4$, $AC = 5$, and $AD = 3$, find the area of $\triangle ABD$.

(a) $2\sqrt{3}$ (b) $\sqrt{11}$ (c) $\sqrt{10}$ (d) 3

12. A five-digit perfect square number $\overline{ABCDE}$, with A and D both nonzero, is such that the two-digit number $\overline{DE}$ divides the three-digit number $\overline{ABC}$. If $\overline{DE}$ is also a perfect square, what is the largest possible value of $\overline{ABC}/\overline{DE}$?

(a) 23 (b) 24 (c) 25 (d) 26

13. Consider the sequence $\{a_n\}$, where $a_1 = 1$, and for $n \geq 2$, we have $a_n = n^{a_{n-1}}$. What is the remainder when a_{2022} is divided by 23?

(a) 11 (b) 12 (c) 21 (d) 22

14. How many ways are there to divide a 5×5 square into three rectangles, all of whose sides are integers? Assume that two configurations which are obtained by either a rotation and/or a reflection are considered the same.

(a) 10 (b) 12 (c) 14 (d) 16

15. Let a_1 be a positive integer less than 200. Define a sequence $\{a_n\}$ by $3a_{n+1} - 1 = 2a_n$ for $n \geq 1$. Let A be the set of all indices m such that a_m is an integer but a_{m+1} is not. What is the largest possible element of A?

(a) 5 (b) 6 (c) 7 (d) 8

PART II. All answers should be in simplest form. Each correct answer is worth five points.

1. Let $S = \{1, 2, ..., 2023\}$. Suppose that for every two-element subset of S, we get the positive difference between the two elements. The average of all of these differences can be expressed as a fraction a/b, where a and b are relatively prime integers. Find the sum of the digits of $a + b$.

2. Let x be the number of six-letter words consisting of three vowels and three consonants which can be formed from the letters of the word "ANTIDERIVATIVE". What is $\lfloor x/1000 \rfloor$?

3. Let $f(x) = \cos(2\pi x/3)$. What is the maximum value of $[f(x+1) + f(x+14) + f(x+2023)]^2$?

4. A function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is defined by $f(0) = 0$ and $f(n) = 1 + f(n - 3^{\lfloor \log_3 n \rfloor})$ for all integers $n \geq 1$. Find the value of $f(10^4)$.

5. Let $\triangle ABC$ be equilateral with side length 6. Suppose P is a point on the same plane as $\triangle ABC$ satisfying $PB = 2PC$. The smallest possible length of segment PA can be expressed in the form $a + b\sqrt{c}$, where a, b, c are integers, and c is not divisible by any square greater than 1. What is the value of $a + b + c$?

6. In chess, a rook may move any number of squares only either horizontally or vertically. In how many ways can a rook from the bottom left corner of an 8×8 chessboard reach the top right corner in exactly 4 moves? (The rook must not be on the top right corner prior to the 4th move.)

7. In acute triangle ABC, points D and E are the feet of the altitudes from points B and C respectively. Lines BD and CE intersect at point H. The circle with diameter DE again intersects sides AB and AC at points F and G, respectively. Lines FG and AH intersect at point K. Suppose that $BC = 25$, $BD = 20$, and $BE = 7$. The length of AK can be expressed as a/b where a and b are relatively prime positive integers. Find $a - b$.

8. Determine the largest perfect square less than 1000 that cannot be expressed as $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor + \lfloor 6x \rfloor$ for some positive real number x.

9. A string of three decimal digits is chosen at random. The probability that there exists a perfect cube ending in those three digits can be expressed as a/b, where a and b are relatively prime positive integers. Find $a + b$.

10. Point D is the foot of the altitude from A of an acute triangle ABC to side BC. The perpendicular bisector of BC meets lines AC and AB at E and P, respectively. The line through E parallel to BC meets line DP at X, and lines AX and BE meet at Q. Given that $AX = 14$ and $XQ = 6$, find AP.

Answers

Part I. (2 points each)

1. A	**6**. D	**11**. B
2. D	**7**. D	**12**. D
3. D	**8**. C	**13**. C
4. A	**9**. B	**14**. B
5. D	**10**. A	**15**. A

Part II. (5 points each)

1. 11 (2024/3)	**6**. 532
2. 42	**7**. 191 (216/25)
3. 3	**8**. 784
4. 8	**9**. 301 (101/200)
5. 11 ($2\sqrt{13} - 4$)	**10**. 35