26th Philippine Mathematical Olympiad

Area Stage, 13 January 2024

PART I. The answer to each item is an integer from 1 to 999. No solution is needed. Write your answers in the answer sheet. Each correct answer is worth three points.

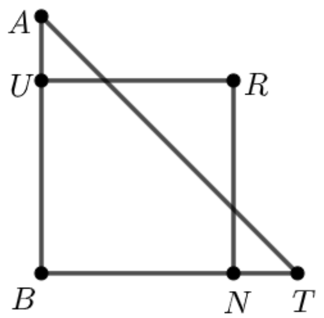
1. Let a and b be relatively prime integers such that $a-b$ and $a+b$ are *not* relatively prime. What is the greatest common divisor of $a-b$ and $a+b$?

2. In the figure below, the square $BURN$ and the isosceles triangle BAT have the same area. What is the measure of $\angle ART$, in degrees?

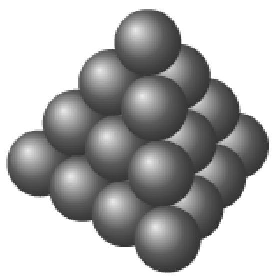
3. The Department of Education of the Philippines orders public schools (elementary or secondary) to have class sizes between 15 and 65, inclusive (*Order Number 62, Series of 2004*). Suppose that all public schools comply with this order, and that each possible class size is attained by some class in some public school. What is the largest integer N with the property that there is certainly some class in some public school in the country that has at least N students with the same birth month?

4. How many ordered pairs of positive integers (x, y) are there so that the set $\{x, y, 1, 3, 5\}$ has a unique mode, and the arithmetic mean, median, and mode have the same value?

5. A stack of balls in the shape of a triangular pyramid contains 2024 balls. How many layers does it have? (Below is an example of a triangular pyramid with 4 layers.)



For Problem 2



For Problem 5

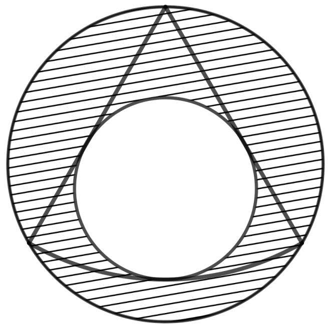
6. What is the greatest number of checkers that can be placed on the squares of a 26×26 checkerboard so that each unit square contains at most one checker, each checker lies entirely inside one unit square, and no four of them lie on the same 2×2 square?

7. Let $f(x) = 1 - 1/x$. Suppose that a, b, and c are the roots of the equation

$$x + f(x) + f(f(x)) = 26.$$

Find $a^2 + b^2 + c^2$.

8. A circular sector has central angle $60°$. A smaller circle is inscribed in the sector, tangent to the two radii and the arc; a larger circle is circumscribed about the same sector as shown below. If the area of the shaded region is equal to 2024, find the remainder when the area of the smaller circle is divided by 1000.



9. Let n be the number of ways all of the letters of the word OLYMPIAD can be placed on the squares of a 3×3 tic-tac-toe board (at most one letter per square) such that no two vowels are on the same column and no two vowels are on the same row. What are the first three digits of n? (Recall that A, E, I, O, U are vowels.)

10. Fern writes down all integers from 1 to 999 that do not contain the digit 2, in increasing order. Stark writes down all integers from 1 to 999 that do not contain the digit 6, in increasing order. How many numbers are written in both of their lists, in the same position?

11. For positive integers p, a, b, and q, let $[p, a, b, q]$ be the sum of the first and last digits of $p^a + b^q$. For example, $[2, 10, 10, 3] = 2 + 4 = 6$ since $2^{10} + 10^3 = 1024 + 1000 = 2024$. Consider the values of $[p, a, 10, q]$ for all p, a, q such that $p \leq 10$, $a \leq 10$, $10 \leq q \leq 20$, and the integers p and q are prime numbers. Give the first three digits of the sum of all these values. (Note that if a value of $[p, a, 10, q]$ appears more than once, then it should be considered in the sum as many times as it appears.)

12. Let $P(x)$ be a polynomial of degree 2 and with leading coefficient 1 such that $P(k) \in \{r, s, t\}$ for $k = 1, 2, 3, 4, 5, 6$, and where r, s, t are real numbers satisfying $r+s+t = 2024$. Find the value of $P(0)$.

13. Let n be the smallest positive integer for which there are exactly 2023 positive integer solutions (x, y) to the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

Determine the first three digits of n.

14. For a set T of numbers, we define $S(T)$ to be the sum of the elements of T. We take $S(\varnothing) = 0$. As usual, $|T|$ denotes the number of elements of T. Let A be the set of positive integers less than 1000 whose digits consist of only 1s or 3s (or both). We choose a random subset B of A. The probability that $S(B)$ and $|B|$ end in the same digit can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

15. How many ways are there to arrange all the digits in the number 629,999,999 so that the resulting number is divisible by 37?

16. In triangle ABC, suppose that $2\angle BAC + \angle ABC = \angle BCA$, $AB = 260$, and $AC = 26$. If BC can be expressed in the form $m\sqrt{n}$, where m and n are integers, and n is square-free, find $m - n$.

17. Let $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$ be a function such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{1}{x(1-x)}.$$

The fractional part of $f(2024)$ can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the remainder when $b - a$ is divided by 1000.

18. If x is an irrational number and k is an integer such that $x^{14} + kx$, $x^3 + 2x^2$, and $x^2 + x$ are all rational numbers, what is the value of k?

19. Himmel has a watch with two hands: a minute hand and an hour hand. Starting at 12:00 pm, he ran 3 laps around a long oval track at a constant rate, and finished before 1:00 pm. After every full lap, he recorded the distance between the tips of the two hands on his watch. In the order they were recorded, these lengths were 11 mm, 16 mm, and 17 mm. The minute hand is longer than the hour hand by $\sqrt{c}$ mm, where c is a positive integer. What is the value of c?

20. There are n points $P_1, P_2, \cdots, P_n$ on the perimeter of a regular octagon $ABCDEFGH$. None of these points is a vertex of the octagon, and the segments $AP_1, AP_2, \cdots, AP_n$ divide the octagon into regions of equal area. Suppose that one of the sides of the octagon contains exacly 4 of the n points, and its opposite side contains exactly 9 of the n points. What is n?

PART II. Write your solutions to each problem in the solution sheets. Each complete and correct solution is worth ten points.

1. Suppose that Eisen can do the following as many times as he wants to a number written on a blackboard: choose two positive integers m and n such that the number on the blackboard is $m \cdot n$, erase this number on the board, and replace it with $m + n$. For example, if 12 is written on the blackboard, Eisen can replace it with 7, because $12 = 3 \times 4$ and $7 = 3 + 4$. If the number 2023 is initially written on a blackboard, determine, with proof, all possible numbers that Eisen can write on the blackboard.

2. Determine all positive integers k less than 2024 for which $4n + 1$ and $kn + 1$ are relatively prime for all integers n.

Answers to the 26th PMO Area Stage

Part I. (3 points each)

1. 2
2. 135
3. 6
4. 3
5. 22
6. 507
7. 630
8. 12 (1012)
9. 259 (25920)
10. 124
11. 976
12. 684
13. 209 ($n = 2^8 \cdot 3^8 \cdot 5^3 = 209,952,000$)
14. 75 ($\frac{11}{64}$)
15. 27
16. 68 ($78\sqrt{10}$)
17. 11 ($\frac{1013}{2024}$)
18. 377
19. 46
20. 54

Part II. (10 points each)

1. The number 2023 is written on a blackboard. Eisen can do the following as many times as he wants: choose two positive integers m and n such that the number on the blackboard is $m \cdot n$, erase the number on the board, and replace it with $m + n$. For example, if 12 is written on the blackboard, Eisen can replace it with 7, because $12 = 3 \times 4$ and $7 = 3 + 4$. Determine, with proof, all possible numbers Eisen can write on the blackboard.

   Answer: Any positive integer greater than or equal to 5.

   Proof. The solution has two parts:

   We first show that all positive integers at least 5 can eventually be written on the blackboard. First, Eisen can write 5 with the following sequence of moves:

   - $2023 = 17 \times 119 \implies 17 + 119 = 136$
   - $136 = 8 \times 17 \implies 8 + 17 = 25$
   - $25 = 5 \times 5 \implies 5 + 5 = 10$
   - $10 = 2 \times 5 \implies 2 + 5 = 7$
   - $7 = 1 \times 7 \implies 1 + 7 = 8$
   - $8 = 2 \times 4 \implies 2 + 4 = 6$
   - $6 = 2 \times 3 \implies 2 + 3 = 5$

   Finally, by choosing $m = 1$ repeatedly, Eisen can always write the number one higher than the one on the blackboard currently. This shows that Eisen can write any positive integer at least 5.

Alternatively, here is an induction solution that shows Eisen can eventually write 5, if the number on the board is at least 5.

Base case: the number on the blackboard is 5. Eisen does not have to do any move at all.

Inductive step: Suppose that if Eisen can write any positive integer at most k, then he can eventually turn it into 5. We show that if the number on the blackboard is instead $k+1$, Eisen can turn it into a positive integer in the range $[5, k]$; by hypothesis, Eisen can then turn this number into 5.

If $k+1$ is even, then choose $m = 2$ and $n = \frac{k+1}{2}$. Because

$$\frac{k+1}{2} + 2 = \frac{k+5}{2} \leq \frac{k+k}{2} = k,$$

the sum is at most k. Otherwise, first choose $m = 1$ and $n = k+1$, turning this number into $k+2$. Then, we choose $m = 2$ and $n = \frac{k+2}{2}$. Because $k+1$ is odd, k must be even, implying $k \geq 6$. Now as

$$\frac{k+2}{2} + 2 = \frac{k+6}{2} \leq \frac{k+k}{2} = k,$$

the sum is at most k again.

We then end similarly to the first solution, taking $m = 1$ repeatedly.

Second, we show that Eisen cannot write any positive integer less than 5. Because we have shown that Eisen can write any positive integer at least 5, another way to phrase this is that there do not exist two positive integers m and n that satisfy both $mn \geq 5$, and $m + n \leq 4$. But the only pairs (m, n) that satisfy the second condition (up to swapping) are:

- $(1, 1)$ with product 1
- $(1, 2)$ with product 2
- $(1, 3)$ with product 3
- $(2, 2)$ with product 4

None of these products are at least 5, so Eisen cannot turn any integer at least 5 into one less than 5. □

2. Determine all positive integers k less than 2024 for which $4n+1$ and $kn+1$ are both relatively prime for all integers n.

Answer: $k = 2, 3, 5, 6, 8, 12, 20, 36, 68, 132, 260, 516, 1028$

Proof. Let $d = \gcd(4n+1, kn+1)$. Since $4n+1$ is odd, d must be odd. Also, $d \mid 4n+1$ and $d \mid kn+1$ for all integers n, so $d \mid n(k-4)$ for all integers n. If we let $n = d+1$, we get $d \mid k-4$. So if $k - 4 = \pm 2^l$ for some integer $l \geq 0$, then d must be a positive odd divisor of $\pm 2^l$, forcing $d = 1$. Thus, all numbers k of the form 4 ± 2^l less than 2024 are valid, which are $2, 3, 5, 6, 8, 12, 20, 36, 68, 132, 260, 516, 1028$.

Now, we will show all other values of k are invalid. If $k \neq 4 \pm 2^l$ for some integer $l \geq 0$, then $k-4$ must have an odd divisor $t > 1$. Note that $2 \mid t-1, t+1$, so $4 \mid t^2 - 1$. Let $n = (t^2-1)/4$, so $4n+1 = t^2$, which is obviously divisible by t, and $kn+1 = 4n+1+(k-4)n = t^2+(k-4)n$, which is divisible by t because $k-4$ is divisible by t. Thus, there exists an integer n such that $\gcd(4n+1, kn+1) > 1$, so such k is invalid. □