26th Philippine Mathematical Olympiad

Qualifying Stage, 02 December 2023

PART I. Choose the best answer. Each correct answer is worth one point.

1. Let a, b, c be positive integers such that

$$3.14 = a + \frac{1}{b + \frac{1}{c}}.$$

What is $a + b + c$?

(a) 11 (b) 13 (c) 17 (d) 19

2. What is the area of a rhombus whose diagonals have lengths 14 and 48, respectively?

(a) 100 (b) 336 (c) 625 (d) 672

3. The arithmetic mean of 11 integers is 10. After adding 20 to each of the first four and subtracting 24 from each of the last seven, what is the new mean?

(a) 2 (b) 3 (c) 4 (d) 6

4. Let a and b be the last two digits of the 5-digit number $\overline{764ab}$. What is the largest possible value of the product of ab^2 if the 5-digit number is divisible by 6?

(a) 576 (b) 567 (c) 512 (d) 441

5. An urn contains two white and two black balls. John draws two balls simultaneously from the urn. If the balls are of different colors, he stops. Otherwise, he returns both balls to the urn and then repeats the process. What is the probability that he stops after exactly three draws?

(a) $\dfrac{1}{16}$ (b) $\dfrac{2}{27}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{4}{27}$

PART II. Choose the best answer. Each correct answer is worth two points.

1. Today — the 2nd day of the 12th month of the year 2023 — marks the start of the 26th Philippine Mathematical Olympiad. What is the remainder when 2023^{12^2} is divided by 26?

(a) 1 (b) 2 (c) 3 (d) 4

2. There are 2024 coins laid out in a row. For some positive integer x, at least $\dfrac{2}{3}$ of the first x coins are heads, and at least $\dfrac{4}{5}$ of the last x coins are tails. What is the maximum possible value of x?

(a) 1012 (b) 1079 (c) 1288 (d) 1380

3. Find the remainder when $\sum_{n=1}^{2023} 2023^n$ is divided by 15.

(a) 4 (b) 7 (c) 10 (d) 13

4. The integers from 1 to 2023 are written on a long blackboard in a straight line. Millie colors any number divisible by 23 in blue, any number divisible by 88 in yellow, any number divisible by both 23 and 88 in green, and any number not divisible by either in red. How many adjacent pairs of numbers have different colors?

(a) 222 (b) 220 (c) 218 (d) 216

5. Let r and s be the roots of the polynomial $x^2 + 2x + 3$. The value of $\frac{1}{r^2 - 1} + \frac{1}{s^2 - 1}$ is

(a) $-\frac{1}{5}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{5}$ (d) 1

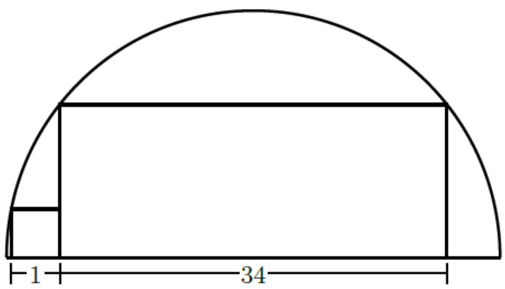
6. Eight people are to sit around a round table on equally-spaced seats. Two of the eight people, Alice and Bob, insist on sitting next to each other. Meanwhile, another two, Clara and Dan, insist on sitting opposite each other. How many ways are there to seat the eight people? Rotations are considered equivalent.

(a) 96 (b) 144 (c) 192 (d) 240

7. A finite sequence $a_1, a_2, \ldots, a_n$, with $n \geq 3$, is said to be *strongly unimodal* if there exists an integer k, with $1 < k < n$, such that $a_1 < \cdots < a_k > a_{k+1} > \cdots > a_n$. How many strongly unimodal sequences $a_1, \ldots, a_{26}$ are there which are permutations of the set $\{1, 2, \ldots, 26\}$?

(a) $2^{25} - 2$ (b) $2^{25} - 1$ (c) $2^{26} - 2$ (d) $2^{26} - 1$

8. A square of side length 1 and a rectangle of length 34 are inscribed in a semicircle as shown below. What is the area of the rectangle?



(a) 204 (b) 238 (c) 272 (d) 306

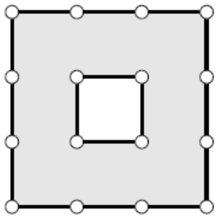
9. A palindrome is a number that reads the same backward and forward. If a palindrome between 100 and 1000 (inclusive) is chosen uniformly at random, what is the probability that this number is divisible by 11?

(a) $\frac{1}{15}$ (b) $\frac{4}{45}$ (c) $\frac{8}{45}$ (d) $\frac{1}{5}$

10. Ernest has a jar with 26 identical cookies. He wants to finish the cookies in the jar by eating 2 or 3 cookies each day. In how many ways can he do this?

(a) 616 (b) 619 (c) 622 (d) 625

11. Below is a drawing of the lattice polygon P:



It has 16 integer points (i.e., points with integer coordinates) on its boundary (indicated by circles), and none in its interior (shaded light gray above).

The polygon $26P$ is obtained by dilating P by a factor of 26. That is, it consists of the points $(26x, 26y)$ wherein (x, y) is a point in the interior or on the boundary of P.

How many integer points are in the interior of $26P$?

(a) 5100 (b) 5200 (c) 5512 (d) 5616

12. The real numbers x, y are such that $x \neq y$ and

$$\frac{x}{26 - x^2} = \frac{y}{26 - y^2} = \frac{xy}{26 - (xy)^2}.$$

What is $x^2 + y^2$?

(a) 626 (b) 650 (c) 677 (d) 729

13. Three lines with slope 1, 7, and c intersect to form a nondegenerate isosceles triangle. If $1 < c < 7$, what is c?

(a) $\frac{8}{7}$ (b) $\frac{13}{7}$ (c) 2 (d) $\frac{15}{7}$

14. Let a_n be a sequence defined by $a_1 = 17$ and $a_{n+1} = \dfrac{2023a_n}{2023 - 2a_n}$. How many terms in the sequence are integers?

(a) 4 (b) 6 (c) 8 (d) 10

15. Let $ABCD$ be a square. Point E is in its interior such that $\angle AED = 90°$ and $\angle EBC = \angle EAB + 45°$. If $AE = 9\sqrt{2}$, find BE.

(a) 9 (b) $9\sqrt{2}$ (c) 18 (d) $18\sqrt{2}$

PART III. The answer to each item is an integer from 1 to 999. Each correct answer is worth five points.

1. What is the largest prime number which divides

$$\frac{2 + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + \cdots + 2023 \cdot 2023!}{2022!}?$$

2. Tasyo is thinking of a two digit number. He tells Basilio the tens digit and Crispin the ones digit, and challenges them to guess the number. Their conversation goes as follows:

   Crispin: I know the number is not prime.

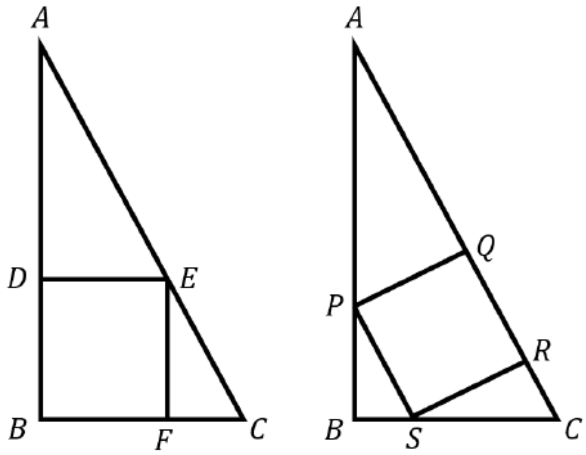
   Tasyo: Correct, it is divisible by 3.

   Basilio: Now I know what the number is.

   What digit did Tasyo tell Crispin?

3. Let $ABCD$ be a square and P be a point on segment AB. Segments CP and BD intersect at Q, and BD is extended beyond B to a point S. Finally, T is the intersection of line CP and the line through S parallel to AB. If $AB = 18$ and $PT = 2CQ$, what is the area of quadrilateral $PBST$?

4. Let ABC be a right triangle with hypotenuse CA. Suppose $AB = 12$ and $BC = 5$. A square is inscribed in this triangle in two different ways, as shown in the figure below.



   The ratio of the length of PQ to that of DE can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

5. For a nonnegative integer n, we define $S(n)$ to be the sum of the decimal digits of n. Define $f(N) = \sum_{n=1}^{N} |S(n) - S(n-1)|$. How many positive integers less than or equal to 2024 are **not** equal to $f(N)$ for some N?

6. Let x_1, x_2, x_3 be real numbers such that $x_k > -2k$, for $k = 1, 2, 3$ and $x_1 + 3x_2 + 5x_3 \leq 28$. The minimum value of
$$\frac{1}{x_1+2} + \frac{3}{x_2+4} + \frac{5}{x_3+6}.$$
   can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

7. For positive integers i, j, k, ℓ less than 5, let $A_{ijk\ell}$ be an integer. Suppose that the integers $A_{ijk\ell}$ satisfy the following properties:

   (a) $A_{ijk\ell} = -A_{jik\ell}$;

   (b) $A_{ijk\ell} = -A_{ij\ell k}$;

   (c) $A_{ijk\ell} = +A_{k\ell ij}$.

   What is the maximum number of distinct integers $A_{ijk\ell}$?

8. If p and q are prime numbers such that p divides $26q - 1$ and q divides $26p + 1$, find the sum of all possible values of $p + q$.

Answers to the 26th PMO Qualifying Stage

Part I. (1 point each)

1. C

2. B

3. A

4. C

5. B

Part II. (2 points each)

1. A

2. D

3. A

4. D

5. B

6. C

7. A

8. A

9. B

10. A

11. B

12. C

13. C

14. D

15. C

Part III. (5 points each)

1. 23

2. 4

3. 648

4. 450 (221/229)

5. 895

6. 17 (9/8)

7. 43

8. 398