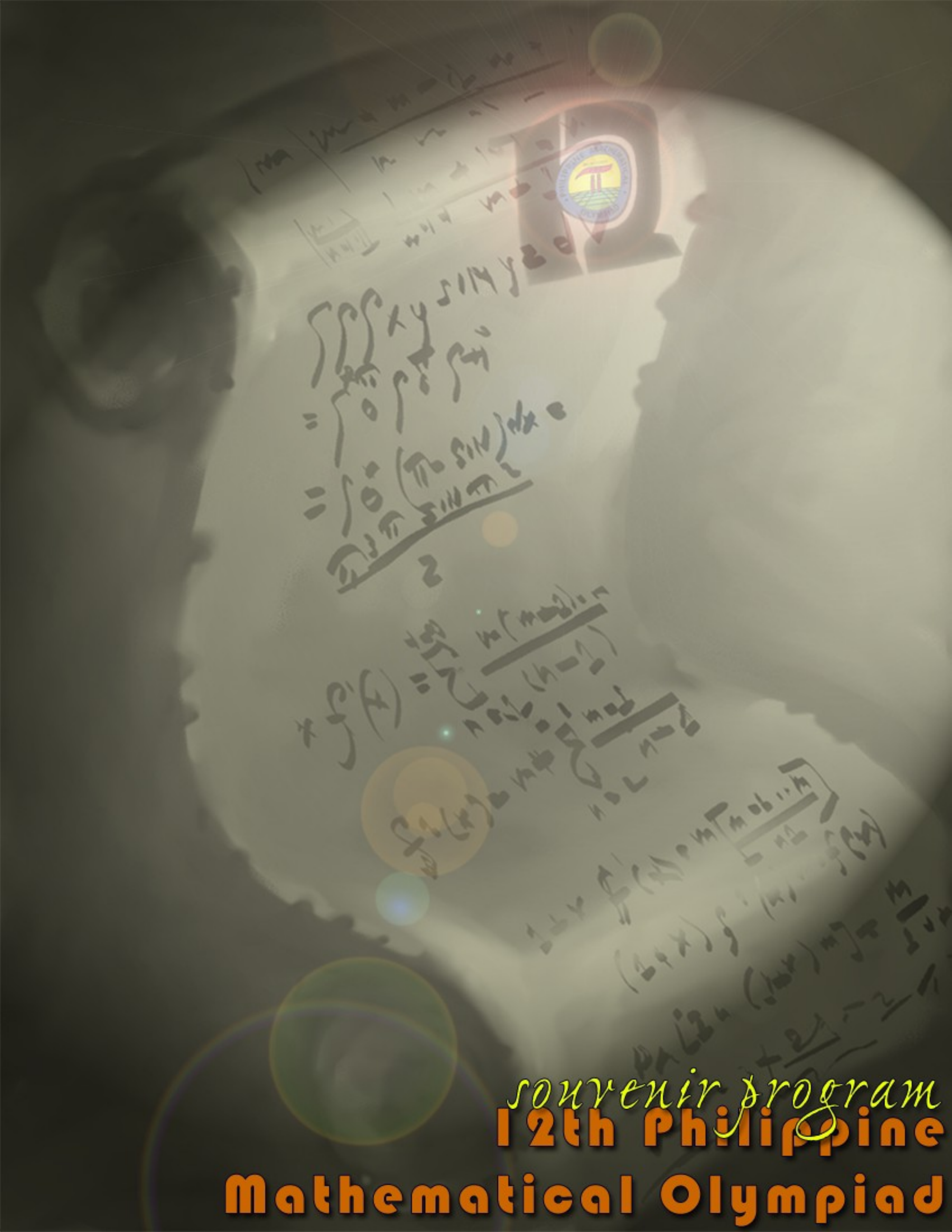


*souvenir program*

# 12th Philippine Mathematical Olympiad

First held in 1984, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: (1) to awaken greater interest in and promote the appreciation of mathematics among students and teachers; (2) to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; (3) to provide a vehicle for the professional growth of teachers; and (4) to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines

The PMO is the first part of the selection process leading to participation in the International Mathematical Olympiad (IMO). It is followed by the Mathematical Olympiad Summer Camp (MOSC), a five-phase program for the twenty national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the twelfth since 1984. Three thousand four hundred five (3405) high school students from all over the country took the qualifying examination, out of these, two hundred twelve (212) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty and these twenty students will compete for the top three positions and hopefully move on to represent the country in the 51st IMO, which will be held in Astana, Republic of Kazakhstan on July 2-14, 2010.

# Message from DOST

The Science Education Institute once again congratulates the Mathematics Society of the Philippines for successfully conducting the 12th Philippine Mathematics Olympiad. The MSP's unwavering commitment to raise the quality of mathematics education in the country through competitions like the PMO.

The country's oldest competition remains formidable in discovering exemplary talent in mathematics. For two years in a row, the PMO has produced medalists to the International Olympiad, the most prestigious mathematics competition in the world. Participants of the PMO bring out their best through alertness, accuracy and discipline as they battle it out to become winners of this competition.

The new decade ushers in with the greater challenge of improving on the past achievements we have achieved. Our back-to-back medals at the IMO pave the way for a higher endeavor of taking mathematics to the next level, a task that SEI fully supports. We truly believe that Filipino students, given the correct training and motivation, can make it big in the international sense.

As competitions like the PMO continue to bring out the best in the Filipino race, we are optimistic that our search for Pinoy talent in science and mathematics would always be fruitful. With that in mind, it is our hope that the students would put their talents into good use and pursue a career in science and mathematics.

We look forward to an exciting PMO and wish all the contestants the best.

**ESTER B. OGENA, PhD**
*Director , DOST-SEI*

# Message from MSP

The Mathematical Society of the Philippines is proud to be part of the Philippine Mathematical Olympiad, the toughest and most prestigious math competition in the country. Now on its 37th year, the MSP promotes mathematics research and education, and is committed to developing mathematical talent among our young high school students. After all, many of its members were once participants in the PMO or the other math competitions the MSP has organized, such as the Metro Manila Math Competition. The experience of competition at the highest levels of problem-solving has surely helped mold several generations of Filipino mathematicians and scientists.

In behalf of the MSP, I wish to thank the DOST-Science Education Institute and the other organizations, institutions and individuals for their continued support and commitment to the PMO.

Congratulations to the winners and all the participants of the 12th PMO!

**Fidel R. Nemenzo**
President, Mathematical Society of the Philippines

# Message from C & E

C&E PUBLISHING INC. is a proud organization engaged in publishing and distribution of quality educational materials. With its 16 branches nationwide, it has remained responsive to the potential growth of the local book industry in terms of continuous expansion on print, digital and online educational products for all levels of discipline.

C&E PUBLISHING INC. continues its unwavering support to endeavors that aid in promoting quality education for Filipino students. It sustains its collaboration with various organizations engaged in projects that aim to elevate our current education system.

C&E PUBLISHING INC. recognizes the relentless efforts of the Philippine Math Olympiad in the promotion and development of Mathematics education in the Philippines. C & E PUBLISHING INC. extends its all-out support to PMO in its advocacy to awaken greater interest in and promote the appreciation of mathematics. As major sponsor, C&E maintains its commitment to help Filipino students in the achievement of quality education by continuously providing topnotch educational products and its unyielding assistance in societal programs.

Congratulations to the organizers of the annual PMO and may you continue to bring pride to our country!

**Mr. John Emyl Eugenio**
VP, Sales and Marketing Division

**C&E Publishing, Inc.**
**Towards Academic and Professional Excellence**

# Message from HSBC

I'd like to express my sincerest congratulations to all the students who have participated in the 2009-2010 Philippine Mathematical Olympiad. Most especially to the top 20 students who will participate in the National Stage this year.

I have always been an avid believer in the value of not only a good math foundation in our Filipino students, but also in the importance of a sound education for the youth. Education capacitates us to see the greater whole: our days in the halls of our schools serve as the springboard for the continuing education we receive from our daily experiences outside of the university. Your talents and skills in the field of Mathematics will inevitably extend onto the hard work and dedication you will need in your lives and careers in the years to come.

I urge you to use your academic aptitude as an endeavor to rouse and stimulate the love of mental adventure; to cultivate in yourselves the ability to find possibilities while transforming yourselves into well-rounded persons. It is important to remember that your responsibility as students should focus on making your lives, through your studies, productive of real value and of helping your fellow students to use their lives in the same manner. I am hopeful that you will all continue to work hard and strive to be excellent not only in Math, but also in everything you undertake from here on end.

Congratulations and the best of luck to you all!

**Jose Arnulfo "WICK" Veloso**
Treasurer and Head of Global Markets,
HSBC Philippines

Foundation for Upgrading the Standard of Education, Inc.

GREETINGS

I am happy to note your efforts and enthusiasm in raising the competencies and skills of Filipino talents in this subject through the

PHILIPPINE MATHEMATICS OLYMPIAD

I am sure your efforts will make an impact on the school curriculum and motivate students to do their best in Mathematics.

**Dr. Lucio C. Tan**
*Vice-Chairman , FUSE*

# Message from Casio

Greetings of Peace and Voyage!

On behalf of CASIO COMPUTER CO., LTD. and Business Plus Marketing, the exclusive distributor of CASIO Calculators and its partner products in the Philippines, I would like to sincerely thank the officers, organizers and all the people behind the $12^{th}$ PHILIPPINE MATHEMATICAL OLYMPIAD especially Ateneo De Manila University and Mathematical Society of the Philippines, for giving us the opportunity to be part of this one educational, essential and scholarly competition in the Mathematics field.

Over the years, Business Plus Marketing (BPM) has been striving hard for the development of Mathematics education among secondary and tertiary schools in the Philippines with the use of CASIO's innovative graphing and scientific calculators. We have built the CASIO MATH SOCIETY (CMS) which already produced numerous (and counting!) educational advancements through seminars and competitive mathematics contests in academes nationwide. These efforts in the Mathematics field are creating a lot of effective and efficient outcome not only in the business division but also in helping our Filipino youth explore and maximize the use of these educational tools in their learning. Through school partnerships such as competitions and seminars, information about CASIO products are widely disseminated and have reached thousands of educators and learners. We are pleased to be working with a prestigious school like yours which is one in our goal to develop technological schools in Mathematic and Science education.

Business Plus Marketing (BPM) and its people are gratified that ADMU together with the faculty, students and participants of the event, put their trust on CASIO. We are looking forward to a successful and beneficial result of the PMO's Math Competition as well as to our strengthened relationship for educational dealings in the future.

More power and May God be with us all the time!

**Joel C. Serrano**
*Sales and Marketing Manager*
*Business Plus Marketing—Casio Philippines*


# PMO 2009-2010

The 12th PMO Team

## THE PMO WORKING TEAM

*Director*
Reginaldo Marcelo

*Assistant Directors*
Joyce Habitan
Stella Monique Salvo
Daniel Andrew Tan

*Test Development Committee*
Alva Benedict Balbuena
Julius Basilla
Evangeline Bautista
Ian Jun Garces (chair)
Job Nable

*Logistics and Operations Committee*
Corazon Alvarado
Editha Bagtas
Karl Friedrich Mina
Ryan Tamayo
Marvin Ticzon

*Awarding Ceremonies Committee*
Eurlyne Domingo
Jumela Sarmiento
Maria Theresa Tulao

## REGIONAL COORDINATORS

*Region 1/CAR*
Ms. Divina Lara

*Region 2*
Mr. Crizaldy P. Binarao

*Region 3*
Dr. Jumar Valdez

*Region 4A*
Dr. Virgilio Sison

*Region 4B*
Engr. Elucila M. Sespeñe

*Region 5*
Dr. Fred Fabay

*Region 6*
Mr. Lindley Kent Faina

*Region 7*
Dr. Lorna Almocera

*Region 8*
Mr. Jonas Villas

*Region 9*
Dr. Paul T. Acebes

*Region 10, 12, ARMM*
Dr. Ferdinand P. Jamil

*Region 11*
Ms. Gina Lapaza– Montalan

*Region 13*
Dr. Thelma C. Montero-Galliguez

*NCR*
Dr. Reginaldo M. Marcelo

# Schedule of PMO National Stage

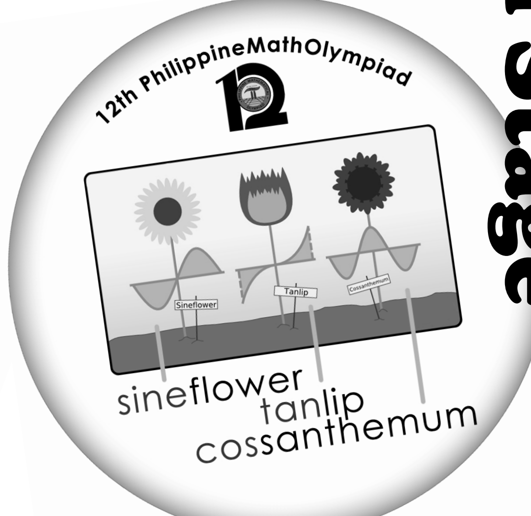
# *Welcome Finalists!*

**Philippine Mathematical Olympiad (PMO)**
National Stage
January 23, 2010

HSBC The world's local bank · FUSE · Business Plus Marketing · CASIO Calculators

| | |
|---|---|
| 7:30 a.m. - 8:30 a.m. | **Registration** |
| 9:00 a.m.– 12:00 p.m. | **Phase I - Written Phase** |
| 12:00 p.m. - 2:00 p.m. | **Lunch Break** |
| 2:00 p.m. - 5:00 p.m. | **Phase II - Oral Phase** |
| | National Anthem |
| | Invocation |
| | Welcome Remarks |
| | Awarding of Certificates |
| | Oral Competition |
| 6:30 p.m. - 8:30 p.m. | **Dinner and Awarding Ceremonies** |





# The 12th PMO National Finalists

Elvis Jeremy D. Ayroso
Philippine Science High School

Geraldine L. Baniqued
St. Paul College, Pasig

Kate Andrea C. Bonamy
Grace Christian College

Kevin Christopher R. Catbagan
Ateneo de Manila High School

Arielle Elise Chua
Saint Jude Catholic School

Lance Robin S. Chua
Bayanihan Institute

Kenneth T. Co
Philippine Science High School

Immanuel V. Encarnacion
Philippine Science High School

Ho Se Kim
La Salle Greenhills

Carmela Antoinette S. Lao
Saint Jude Catholic School

Henry Jefferson C. Morco
Chiang Kai Shek College

Samuel Christian C. Ong
Uno High School

Aldric Cristoval C. Reyes
Chiang Kai Shek College

Amiel S. Sy
Philippine Science High School

Stephen Jann A. Tamayo
Philippine Science High School

Emiliano M. Tan
Philippine Science High School

Ralph Ryan C. Ting
Jubilee Christian Academy

John Russell P. Virata
Gideon Academy

Zheng Rong S. Wu
Zamboanga Chong Hua High School

Zixin Zhang
Grace Christian College
(High School Division)

# The Questions

## Qualifying Stage 1 (Oct. 24,2009)

**1.** If $\underbrace{2009 + 2009 + \cdots + 2009}_{2009 \text{ terms}} = 2009^x$, find the value of $x$.

(a) 2 (b) 3 (c) 2009 (d) 2010

**2.** What is the least positive difference between two three-digit numbers if one number has all the digits 2, 4, and 6, while the other has all the digits 1, 2, and 9?

(a) 72 (b) 54 (c) 48 (d) 27

**3.** Which of the following numbers is closest to one of the roots of the equation $x^2 - 10000x - 10000 = 0$?

(a) 10001 (b) 5000 (c) 100 (d) 25

**4.** The ratio of the areas of two squares is 3 : 4. What is the ratio of the lengths of their corresponding diagonals?

(a) $1 : 2$ (b) $3 : 4$ (c) $2 : 3$ (d) $\sqrt{3} : 2$

**5.** In $\triangle ABC$, let $P$ be a point on segment $BC$ such that $BP : PC = 1 : 4$. Find the ratio of the area of $\triangle ACP$ to that of $\triangle ABC$.

(a) $1 : 4$ (b) $1 : 5$ (c) $3 : 4$ (d) $4 : 5$

**6.** In how many ways can three distinct numbers be selected from the set $\{1, 2, 3, \ldots, 9\}$ if the product of these numbers is divisible by 21?

(a) 15 (b) 16 (c) 17 (d) 18

**7.** If $|2x - 3| \le 5$ and $|5 - 2y| \le 3$, find the least possible value of $x - y$.

(a) $-5$ (b) 0 (c) $-1$ (d) 5

**8.** Define the operations $\clubsuit$ and $\heartsuit$ by

$$a \clubsuit b = ab - a - b \quad \text{and} \quad a \heartsuit b = a^2 + b - ab.$$

What is the value of $(-3 \heartsuit 4) - (-3 \clubsuit 4)$?

(a) 38 (b) 12 (c) $-12$ (d) $-38$

**9.** Solve for $x$ in the following system of equations:

$$\begin{cases} \log x + \log y &= 2 \\ \log y + \log z &= 7 \\ \log z + \log x &= 3. \end{cases}$$

(a) 10 (b) 1 (c) 0.1 (d) 0.01

**10.** If $a + 1 = b - 2 = c + 3 = d - 4$, which is the smallest among the numbers $a$, $b$, $c$, and $d$?

(a) $a$ (b) $b$ (c) $c$ (d) $d$

**11.** Solve for $x$ in the inequality $5^x \ge 25^{2x}$.

(a) $x \le 1$ (b) $x \le 0$ (c) $x \ge 0$ (d) $x \ge 1$

**12.** Today, 24 October 2009, is a Saturday. On what day of the week will 10001 days from now fall?

(a) Sat (b) Mon (c) Thurs (d) Fri

**13.** The lines $2x + ay + 2b = 0$ and $ax - y - b = 1$ intersect at the point $(-1, 3)$. What is $2a + b$?

(a) $-6$ (b) $-4$ (c) 4 (d) 6

**14.** Let $x$ be a real number that satisfies the equation

$$16 (\log_9 x)^4 = \left(\log_3 x^3\right)^2 + 10.$$

Determine $(\log_9 x)^2$.

(a) 10 (b) $\sqrt{10}$ (c) $\frac{5}{2}$ (d) $\frac{\sqrt{5}}{2}$

**15.** Let $r$ and $s$ be the roots of the equation $x^2 - 2mx - 3 = 0$. If $r + s^{-1}$ and $s + r^{-1}$ are the roots of the equation $x^2 + px - 2q = 0$, what is $q$?

(a) 1 (b) $\frac{2}{3}$ (c) $-3$ (d) $-\frac{4}{3}$

**16.** On the blackboard, 1 is initially written. Then each of ten students, one after another, erases the number he finds on the board, and write its double plus one. What number is erased by the tenth student?

(a) $2^{11} - 1$ (b) $2^{11} + 1$ (c) $2^{10} - 1$ (d) $2^{10} + 1$

**17.** For how many real numbers $x$ is $\sqrt{2009-\sqrt{x}}$ an integer?

(a) 0 (b) 45 (c) 90 (d) 2009

**18.** How many distinct natural numbers less than 1000 are multiples of 10, 15, 35, or 55?

(a) 145 (b) 146 (c) 147 (d) 148

**19.** Let $x$ and $y$ be nonnegative real numbers such that $2^{x+2y}=8\sqrt{2}$. What is the maximum possible value of $xy$?

(a) $8\sqrt{2}$ (b) 49/4 (c) 49/32 (d) 1

**20.** In how many ways can ten people be divided into two groups?

(a) 45 (b) 511 (c) 637 (d) 1022

**21.** Let $P$ be the point inside the square $ABCD$ such that $\triangle PCD$ is equilateral. If $AP = 1$ cm, what is the area of the square?

(a) $3+\sqrt{3}$ cm$^2$ (c) $\frac{9}{4}$ cm$^2$
(b) $2+\sqrt{3}$ cm$^2$ (d) 2 cm$^2$

**22.** Let $x$ and $y$ be real numbers such that $2^{2x}+2^{x-y}-2^{x+y}=1$. Which of the following equations is always true?

(a) $x+y=0$ (c) $x+2y=0$
(b) $x=2y$ (d) $x=y$

**23.** In $\triangle ABC$, $M$ is the midpoint of $BC$, and $N$ is the point on the bisector of $\angle BAC$ such that $AN \perp NB$. If $AB = 14$ and $AC = 19$, find $MN$.

(a) 1 (b) 1.5 (c) 2 (d) 2.5

**24.** Seven distinct integers are randomly chosen from the set $\{1, 2, \ldots, 2009\}$. What is the probability that two of these integers have a difference that is a multiple of 6?

(a) $\frac{7}{2009}$ (b) $\frac{2}{7}$ (c) $\frac{1}{2}$ (d) 1

**25.** A student on vacation for $d$ days observed that (1) it rained seven times, either in the morning or in the afternoon, (2) there were five clear afternoons, and (3) there were six clear mornings. Determine $d$.

(a) 7 (b) 8 (c) 9 (d) 10

**26.** How many sequences containing two or more consecutive positive integers have a sum of 2009?

(a) 3 (b) 4 (c) 5 (d) 6

**27.** In $\triangle ABC$, let $D$, $E$, and $F$ be points on the sides $BC$, $AC$, and $AB$, respectively, such that $BC = 4CD$, $AC = 5AE$, and $AB = 6BF$. If the area of $\triangle ABC$ is 120 cm$^2$, what is the area of $\triangle DEF$?

(a) 60 cm$^2$ (c) 62 cm$^2$
(b) 61 cm$^2$ (d) 63 cm$^2$

**28.** A function $f$ is defined on the set of positive integers by $f(1) = 1$, $f(3) = 3$, $f(2n) = n$, $f(4n+1) = 2f(2n+1) - f(n)$, and $f(4n+3) = 3f(2n+1) - 2f(n)$ for all positive integers $n$. Determine

$$\sum_{n=1}^{10}[f(4n+1)+f(2n+1)-f(4n+3)].$$

(a) 55 (b) 50 (c) 45 (d) 40

**29.** A sequence of consecutive positive integers beginning with 1 is written on the blackboard. A student came along and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What number was erased?

(a) 7 (b) 8 (c) 9 (d) 10

**30.** Let $M$ be the midpoint of the side $BC$ of $\triangle ABC$. Suppose that $AB = 4$ and $AM = 1$. Determine the smallest possible measure of $\angle BAC$.

(a) 60° (b) 90° (c) 120° (d) 150°

## Qualifying Stage for Regions I, II, CAR (Nov. 7, 2009)

**1.** If $x \in \mathbb{R}$ and $x^2 - 2x - 2 \geq 0$, what is the least value of $x^2$?

(a) $1-\sqrt{3}$ (c) $4-2\sqrt{3}$
(b) $1+\sqrt{3}$ (d) $4+2\sqrt{3}$

**2.** How many points $(x, y)$ in the $xy$-plane, with positive rational coordinates, satisfy the inequality $6x + 5y \leq 30$?

(a) 8 (c) 10
(b) 9 (d) infinite

**3.** Find all values of $x$ that satisfy the equation $\log_3 2\sqrt{2x-3} > 0$.

(a) $x > 13/8$ (c) $x > 5/2$
(b) $x > 3/2$ (d) $x > 7/8$

**4.** Let $V$ be one of the vertices of a fair die. If the die is tossed on a table, what is the probability that vertex $V$ will be in contact with the table?

(a) $\frac{1}{6}$ (c) $\frac{1}{2}$
(b) $\frac{1}{3}$ (d) $\frac{2}{3}$

**5.** For what values of $a$ does equation $(x-a)^2+(x^2-3x+2)^2 = 0$ have a real solution?

(a) 0 (c) 1 or 2
(b) 1 (d) 0, 1, or 2

**6.** Let $x$ and $y$ satisfy the equation $2y^2 - 2\sqrt{2}y - x - 6 = 0$. For what real values of $x$ is $y$ real?

(a) none (c) $x \leq -6$
(b) $x \geq -7$ (d) all reals

**7.** In how many ways can the letters of the word OLYMPIAD be rearranged in such a way that the vowels appear in alphabetical order and the consonants also appear in alphabetical order?

(a) $3! \cdot 5!$ (b) 816 (c) $\binom{6}{3}$ (d) 56

**8.** How many real values of $x$ does the equation $\sqrt{x+1} = x - 1$ have?

(a) zero (b) one (c) two (d) three

**9.** If $f(x) = 2 \cdot 3^x$, which of the following is equal to $f(1+x) - f(x)$?

(a) $2f(x)$ (c) $-3f(x)$
(b) $(1+x)f(x)$ (d) $f(1)$

**10.** If $3^x = 2^z$ and $3^y = 2^x$, which of the following inequalities is true?

(a) $x < y < z$ (c) $y \leq z < x$
(b) $x < z < y$ (d) $y < x \leq z$

**11.** How many integral values of $n$ will make $35 - 12n + n^2$ a prime?

(a) none (c) two
(b) one (d) infinite

**12.** A real number $x$ satisfies the inequality $-y < x < y$ for all positive real numbers $y$. Which of the following is true?

(a) $x < \frac{1}{2}$ (c) $x = 1$
(b) $\frac{1}{2} \leq x < 1$ (d) $x > 1$

**13.** How many real roots does the equation $\Big|\big||x|+3\big|-2\Big| = 0$ have?

(a) none (b) one (c) two (d) three

**14.** The time required for each of the two examinees to solve any problem differs by 2 minutes. Together, they can solve 32 problems in an hour. How many minutes will it take the slower examinee to solve a problem?

(a) 3 (b) 4 (c) 5 (d) 6

**15.** The smallest and the largest interior angles of a convex polygon measure 90° and 180°, respectively. If the measures of its interior angles are in arithmetic progression, how many sides does the polygon have?

(a) 6 (b) 7 (c) 8 (d) 9

The Qualifying and Area Stages

**16.** Three lines parallel to a base of a triangle divide the other sides into four congruent segments. As a consequence, the area of the triangle is divided into four parts with unequal areas. If the area of the largest part is 28 cm$^2$, what is the area (in cm$^2$) of the original triangle?

(a) 52.5 (b) 56 (c) 64 (d) 98

**17.** Let $T = 3 + 6 + 9 + \cdots + 3N$, where $N$ is the least integer such that $T \geq 2009$. What is $1+2+3+\cdots+N$?

(a) 600 (b) 703 (c) 741 (d) 2109

**18.** A square is inscribed inside an equilateral triangle, where two vertices of the square are on one side of the triangle. Knowing that one side of the triangle is 1 cm long, how long (in cm) is one side of the square?

(a) $\frac{1}{2}$ (c) $\frac{1}{3}$

(b) $2\sqrt{3}-3$ (d) $\sqrt{3}-1$

**19.** Three of the roots of the equation $2x^4 + ax^2 + bx + c = 0$ are $-1$, 2, and $-3$. What is $b$?

(a) 0 (b) 6 (c) 8 (d) $-18$

**20.** Find the area (in square units) of the closed region in the $xy$-plane that is bounded by the $x$-axis, the line $x = 6$, and the graph of the function $f$ defined by

$$y = f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 2 \\ -\frac{1}{2}x + 5 & \text{if } 2 \leq x \leq 6. \end{cases}$$

(a) 4 (b) 8 (c) 16 (d) 32

**21.** Circles $O_1$ and $O_2$ are externally tangent at point $P$, and have radii 2 cm and 4 cm, respectively. Points $A$ and $B$, both different from $P$, are chosen on $O_1$ and $O_2$, respectively, such that $A$, $P$, and $B$ are collinear. If $AB = 4$ cm, how long is $PB$?

(a) 2 cm (c) $\frac{10}{3}$ cm

(b) $1+\sqrt{3}$ cm (d) $\frac{8}{3}$ cm

**22.** What is the remainder when $x^{2009}$ is divided by $x^2 - 1$?

(a) 1 (b) $x$ (c) $x+1$ (d) $x-1$

**23.** One side of square $ABCD$ is 3 cm long. Let $E$ and $F$ be two points inside $ABCD$ such that $EF = 1$ cm, $EF \| AB$, and $A$ is nearer to $E$ than $F$. Find the area of the non-convex hexagon $ABFCDE$.

(a) 5 cm$^2$ (c) 6 cm$^2$

(b) 5.5 cm$^2$ (d) 6.5 cm$^2$

**24.** For how many positive integers $a$ less than 100 does the system

$$\begin{cases} x^2 = y + a \\ y^2 = x + a \end{cases}$$

have integral solutions $(x, y)$?

(a) 19 (b) 20 (c) 9 (d) 10

**25.** Let $O$ be the center of a circle, and let $BA$ and $BC$ be the tangents from a point $B$ to the circle at $A$ and $C$, respectively. The smaller (circular) sector $AOC$ is cut to form a right-circular cone. If $OB = 4$ cm and $\angle ABC = 60°$, determine the base radius of the cone.

(a) $\frac{2}{3}$ cm (c) $\sqrt{2}$ cm

(b) $\frac{\sqrt{3}}{2}$ cm (d) 2 cm

**26.** In $\triangle ABC$, let $P$ and $Q$ be the midpoints of sides $AB$ and $AC$, respectively. Let $BQ$ and $CP$ intersect at the point $G$. If the area of $\triangle AGP$ is 3 square inches, what is the area of $\triangle ABC$ in square inches?

(a) 24 (b) 18 (c) $9\sqrt{2}$ (d) $15\sqrt{3}$

**27.** A positive integer is called *triangular* if it is equal to $\frac{1}{2}n(n+1)$ for some integer $n$. How many pairs $(a, b)$ of triangular numbers are there such that $b - a = 2009$?

(a) four (b) five (c) six (d) seven

**28.** On an island, there are only two kinds of people: those who are always honest, and those who always lie. Three inhabitants are talking. (1) Andrew says "Bernie is honest." (2) Bernie says "Andrew and Cholo are both honest." (3) Cholo says "Andrew is a liar." Which of the following conclusions is correct?

(a) All three inhabitants are honest.
(b) Andrew and Bernie are honest, and Cholo is a liar.
(c) Andrew is honest, and Bernie and Cholo are liars.
(d) Andrew and Bernie are liars, and Cholo is honest.

**29.** Fifty distinct numbers, whose sum is 3000, are chosen at random from the set $\{1, 2, 3, \ldots, 100\}$. What is the least number of even numbers among these fifty numbers?

(a) 3 (b) 4 (c) 5 (d) 6

**30.** Simplify the following expression:

$$\frac{\sqrt{10+\sqrt{1}}+\sqrt{10+\sqrt{2}}+\sqrt{10+\sqrt{3}}+\cdots+\sqrt{10+\sqrt{99}}}{\sqrt{10-\sqrt{1}}+\sqrt{10-\sqrt{2}}+\sqrt{10-\sqrt{3}}+\cdots+\sqrt{10-\sqrt{99}}}.$$

(a) $1+\sqrt{2}$ (c) $2\sqrt{10}$
(b) $\sqrt{20}$ (d) $2+\sqrt{10}$

## Area Stage (Nov. 21, 2009)

**1.** If $a = 2^{-1}$ and $b = \frac{2}{3}$, what is the value of $(a^{-1}+b^{-1})^{-2}$?

**2.** Find the sum of all (numerical) coefficients in the expansion of $(x+y+z)^3$.

**3.** A circle has radius 4 units, and a point $P$ is situated outside the circle. A line through $P$ intersects the circle at points $A$ and $B$. If $PA = 4$ units and $PB = 6$ units, how far is $P$ from the center of the circle?

**4.** Let $y = (1+e^x)(e^x-6)^{-1}$. If the values of $x$ run through all real numbers, determine the values of $y$.

**5.** The sum of the product and the sum of two integers is 95. The difference between the product and the sum of these integers is 59. Find the integers.

**6.** Let $A$, $B$, $C$, $D$ (written in the order from left to right) be four equally-spaced collinear points. Let $\omega$ and $\omega'$ be the circles with diameters $AD$ and $BD$, respectively. A line through $A$ that is tangent to $\omega'$ intersects $\omega$ again at point $E$. If $AB = 2\sqrt{3}$ cm, what is $AE$?

**7.** A certain high school offers its students the choice of two sports: football and basketball. One fifth of the footballers also play basketball, and one seventh of the basketball players also play football. There are 110 students who practice exactly one of the sports. How many of them practice both sports?

**8.** Simplify: $\sqrt{\sin^4 15^\circ + 4\cos^2 15^\circ} - \sqrt{\cos^4 15^\circ + 4\sin^2 15^\circ}$.

**9.** Let $a$, $b$, and $c$ be the roots of the equation $2x^3 - x^2 + x + 3 = 0$. Find the value of

$$\frac{a^3-b^3}{a-b} + \frac{b^3-c^3}{b-c} + \frac{c^3-a^3}{c-a}.$$

**10.** In $\triangle ABC$, let $D$, $E$, and $F$ be points on sides $BC$, $CA$, and $AB$, respectively, so that the segments $AD$, $BE$, and $CF$ are concurrent at point $P$. If $AF : FB = 4 : 5$ and the ratio of the area of $\triangle APB$ to that of $\triangle APC$ is $1 : 2$, determine $AE : AC$.

**11.** A circle of radius 2 cm is inscribed in $\triangle ABC$. Let $D$ and $E$ be the points of tangency of the circle with the sides $AC$ and $AB$, respectively. If $\angle BAC = 45^\circ$, find the length of the minor arc $DE$.

**12.** Two regular polygons with the same number of sides have sides 48 cm and 55 cm in length. What is the length of one side of another regular polygon with the same number of sides whose area is equal to the sum of the areas of the given polygons?

**13.** The perimeter of a right triangle is 90 cm. The squares of the lengths of its sides sum up to 3362 $\text{cm}^2$. What is the area of the triangle?

**14.** Determine all real solutions $(x, y, z)$ of the following system of equations:

$$\begin{cases} x^2 - y = z^2 \\ y^2 - z = x^2 \\ z^2 - x = y^2. \end{cases}$$

**15.** For what value(s) of $k$ will the lines $2x + 7y = 14$ and $kx - y = k + 1$ intersect in the first quadrant?

**16.** For what real numbers $r$ does the system of equations

$$\begin{cases} x^2 &= y^2 \\ (x-r)^2 + y^2 &= 1 \end{cases}$$

have no solutions?

**17.** Determine the smallest positive integer $n$ such that $n$ is divisible by 20, $n^2$ is a perfect cube, and $n^3$ is a perfect square.

**18.** Find all pairs $(a, b)$ of integers such that $\sqrt{2010 + 2\sqrt{2009}}$ is a solution of the quadratic equation $x^2 + ax + b = 0$.

**19.** Determine all functions $f : (0, +\infty) \to \mathbb{R}$ such that $f(2009) = 1$ and

$$f(x)f(y) + f\left(\tfrac{2009}{x}\right) f\left(\tfrac{2009}{y}\right) = 2f(xy)$$

for all positive real numbers $x$ and $y$.

**20.** Find all pairs $(k, r)$, where $k$ is an integer and $r$ is a rational number, such that the equation $r(5k - 7r) = 3$ is satisfied.

**21.** Each of the integers $1, 2, 3, \ldots, 9$ is assigned to each vertex of a regular 9-sided polygon (that is, every vertex receives exactly one integer from $\{1, 2, \ldots, 9\}$, and two vertices receive different integers) so that the sum of the integers assigned to any three consecutive vertices does not exceed some positive integer $n$. What is the least possible value of $n$ for which this assignment can be done?

**22.** Let $E$ and $F$ be points on the sides $AB$ and $AD$ of a convex quadrilateral $ABCD$ such that $EF$ is parallel to the diagonal $BD$. Let the segments $CE$ and $CF$ intersect $BD$ at points $G$ and $H$, respectively. Prove that if the quadrilateral $AGCH$ is a parallelogram, then so is $ABCD$.

**23.** Let $p$ be a prime number. Let $a$, $b$, and $c$ be integers that are divisible by $p$ such that the equation $x^3 + ax^2 + bx + c = 0$ has at least two different integer roots. Prove that $c$ is divisible by $p^3$.

# The Answers

## Qualifying Stage (Oct 24, 2009)

| | | | | |
|---|---|---|---|---|
| **1.** a | **7.** a | **13.** d | **19.** c | **25.** c |
| **2.** d | **8.** a | **14.** c | **20.** b | **26.** c |
| **3.** a | **9.** c | **15.** b | **21.** b | **27.** b |
| **4.** d | **10.** c | **16.** c | **22.** d | **28.** d |
| **5.** d | **11.** b | **17.** b | **23.** d | **29.** a |
| **6.** d | **12.** c | **18.** c | **24.** d | **30.** d |

## Qualifying Stage for Regions I, II, CAR

| | | | | |
|---|---|---|---|---|
| **1.** c | **7.** d | **13.** a | **19.** c | **25.** a |
| **2.** d | **8.** b | **14.** c | **20.** c | **26.** b |
| **3.** a | **9.** a | **15.** c | **21.** d | **27.** c |
| **4.** c | **10.** d | **16.** c | **22.** b | **28.** d |
| **5.** c | **11.** c | **17.** b | **23.** c | **29.** d |
| **6.** b | **12.** a | **18.** b | **24.** a | **30.** a |

## Area Stage

**1.** $\frac{4}{49}$

**2.** $3^3$

**3.** $2\sqrt{10}$ units

**4.** $(-\infty, -\frac{1}{6}) \cup (1, +\infty)$

**5.** 11 and 7

**6.** 9 cm

**7.** 11 students

**8.** $\frac{1}{2}\sqrt{3}$

**9.** $-1$

**10.** $2:7$

**11.** $\frac{3}{2}\pi$ cm

**12.** 73 cm

**13.** 180 cm$^2$

**14.** $(0,0,0), (1,0,-1), (0,-1,1), (-1,1,0)$

**15.** $(-\infty, -3) \cup (\frac{1}{6}, +\infty)$

**16.** $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$

**17.** 1000000

**18.** $(-2, -2008)$

**19.** $f(x) = 1$ for all $x \in (0, +\infty)$

**20.** $(2,1), (-2,-1), (2, \frac{3}{7}), (-2, -\frac{3}{7})$

**Problem 21.** There is an assignment of the integers $1, 2, 3, \ldots, 9$ to the vertices of the regular nonagon that gives $n = 16$. Let $S$ be the sum of all sums of the integers assigned to three consecutive vertices. If there are integers assigned to three consecutive vertices whose sum is at most 14, then $S < 135$, which is a contradiction. Thus, every sum of the integers assigned to three consecutive vertices is equal to 15. Consider $a, b, c, d$. Then $a + b + c = 15$ and $b + c + d = 15$, which implies that $a = d$, a contradiction again. □

**Problem 22.** Let $EF$ intersect $AG$ and $AH$ at points $I$ and $J$, respectively. Note that $EGHJ$ and $FIGH$ are parallelograms. It follows that $\triangle FHJ \cong \triangle IGE$, which implies that $FJ = EI$. Using three pairs of similar triangles, $FJ = EI$ implies that $BG = DH$.

Let $S$ be the midpoint of $AC$. Since $AGCH$ is parallelogram, $S$ is the midpoint of $GH$. Finally, with $BG = DH$, we have $BS = BG + GS = DH + HS = DS$. This means that the diagonals $AC$ and $BD$ bisect each other, and so $ABCD$ is a parallelogram. □

**Problem 23.** Let $r$ and $s$ be two different integral roots of $x^3 + ax^2 + bx + c = 0$; that is, $r^3 + ar^2 + br + c = 0$ and $s^3 + as^2 + bs + c = 0$. Since $p$ divides $a$, $b$, and $c$, it follows that $p$ divides both $r^3$ and $s^3$. Being prime, $p$ divides $r$ and $s$.

Subtracting the above equations involving $r$ and $s$, we get

$$r^3 - s^3 + a(r^2 - s^2) + b(r - s) = 0,$$

or

$$(r - s)(r^2 + rs + s^2 + a(r + s) + b) = 0.$$

Since $r \neq s$, the last equation becomes

$$r^2 + rs + s^2 + a(r + s) + b = 0.$$

Because the terms (other than $b$) are divisible by $p^2$, the last equation forces $p^2$ to divide $b$.

Finally, the terms (other than $c$) of $r^3 + ar^2 + br + c = 0$ are divisible by $p^3$, it follows that $p^3$ divides $c$. □

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