26^{th} Philippine Mathematical Olympiad

National Stage (Day 1)

17 February 2024

Time: 4.5 hours *Each item is worth 7 points.*

1. Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be a function satisfying

$$f(x+1, y) + f(x, y+1) + 1 = f(x, y) + f(x+1, y+1)$$

for all integers x and y. Can it happen that $|f(x,y)| \leq 2024$ for all $x, y \in \mathbb{Z}$?

2. Let $0!! = 1!! = 1$ and $n!! = n \cdot (n-2)!!$ for all integers $n \geq 2$. Find all positive integers n such that

$$\frac{(2^n + 1)!! - 1}{2^{n+1}}$$

is a positive integer.

3. Given triangle ABC with orthocenter H, the lines through points B and C that are perpendicular to lines AB and AC respectively, intersect line AH at points X and Y respectively. The circle with diameter XY intersects lines BX and CY a second time at points K and L respectively. Prove that points H, K, and L are collinear.

4. Let n be a positive integer. For any $\mathcal{S} \subseteq \{1, 2, \cdots, n\}$, let $f(\mathcal{S})$ be the set containing all positive integers at most n that have an odd number of factors in $\mathcal{S}$. How many subsets of $\{1, 2, \cdots, n\}$ can be turned into $\{1\}$ after finitely many (possibly zero) applications of f?

26^{th} Philippine Mathematical Olympiad

National Stage (Day 2)

18 February 2024

Time: 4.5 hours *Each item is worth 7 points.*

5. Find the largest positive integer k so that any binary string of length 2024 contains a palindromic substring of length at least k.

6. For a real number x, let $\lfloor x \rfloor$ be the greatest integer not exceeding x. The sequence $\{a_n\}_{n\geq 1}$ of real numbers is defined as follows:

$$a_1 = 1, \text{ and } a_{n+1} = \frac{1}{2\lfloor a_n \rfloor - a_n + 1} \text{ for all } n \geq 1.$$

Find, with proof, the value of a_{2024}.

7. Let ABC be an acute triangle with orthocenter H, circumcenter O, and circumcircle Ω. Points E and F are the feet of the altitudes from B to AC and C to AB, respectively. Let line AH intersect Ω again at point $D \neq A$. The circumcircle of DEF intersects Ω again at X, and AX intersects BC at I. The circumcircle of triangle IEF intersects BC again at G. If M is the midpoint of BC, prove that lines MX and OG intersect at a point on Ω.

8. Let $\varphi(n)$ denote the number of positive integers $m \leq n$ satisfying $\gcd(m, n) = 1$. Find all positive integers n for which $\varphi(\varphi(n))$ divides n.

Solutions to the 26th Philippine Mathematical Olympiad

1. Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be a function satisfying

$$f(x+1, y) + f(x, y+1) + 1 = f(x, y) + f(x+1, y+1)$$

for all integers x and y. Can it happen that $|f(x, y)| \leq 2024$ for all $x, y \in \mathbb{Z}$?

<u>*Solution.*</u> We prove by contradiction; suppose that $|f(x, y)| \leq 2024$ is true.

Claim: $f(x, y) = f(x, 0) + f(0, y) - f(0, 0) + xy$ for all integers x, y.

Proof: We induct on $|x + y|$. The base cases are $|x + y| = 0$ or $|x + y| = 1$, where clearly the equation is true because either $x = 0$ or $y = 0$ (as all terms but one in the RHS cancel out). For the inductive step, suppose $x, y > 0$ (the other cases work similarly). Then we have

$$\begin{aligned} f(x, y) =& f(x-1, y) + f(x, y-1) - f(x-1, y-1) + 1 \\ =& (f(x-1, 0) + f(0, y) - f(0, 0) + (x-1)(y)) \\ &+ (f(x, 0) + f(0, y-1) - f(0, 0) + (x)(y-1)) \\ &- (f(x-1, 0) + f(0, y-1) - f(0, 0) + (x-1)(y-1)) + 1 \\ =& f(x, 0) + f(0, y) - f(0, 0) + xy \end{aligned}$$

as desired.

Now, let $x = y = M$ for some positive integer M. We now have

$$\begin{aligned} f(M, M) &= f(M, 0) + f(0, M) - f(0, 0) + M^2 \\ \implies f(M, M) - f(M, 0) - f(0, M) &= M^2 - f(0, 0) \\ \implies |f(M, M) - f(M, 0) - f(0, M)| &= M^2 - f(0, 0) \end{aligned}$$

by taking M large enough so the RHS becomes positive. In fact, because squares are unbounded above, we can always choose a large enough M so that $M^2 - f(0, 0) > 6072$. But then by the triangle inequality

$$|f(M, M) - f(M, 0) - f(0, M)| \leq |2024| + |2024| + |2024| = 6072,$$

which is a contradiction. ■

2. Let $0!! = 1!! = 1$ and $n!! = n \cdot (n-2)!!$ for all integers $n \geq 2$. Find all positive integers n such that

$$\frac{(2^n+1)!!-1}{2^{n+1}}$$

is a positive integer.

Solution. We claim that all integers $n \geq 3$ work.

Note that $n = 1$ and $n = 2$ give us $\dfrac{(2^n+1)!!-1}{2^{n+1}} = \dfrac{1}{2}$ and $\dfrac{(2^n+1)!!-1}{2^{n+1}} = \dfrac{7}{4}$ respectively, which are not integers. So $n \geq 3$.

We claim that all $n \geq 3$ work. Note that 2^{n-2} is an even integer. Also, note that $5^{2^{n-1}} = (1+4)^{2^{n-1}} \equiv 1 \pmod{2^{n+1}}$, so the order of 5 $\pmod{2^{n+1}}$ must divide 2^{n-1}. We see that $5^{2^{n-2}} = (1+4)^{2^{n-2}} \equiv 1 + 2^n \pmod{2^{n+1}}$, so 2^{n-1} must be the order of 5 $\pmod{2^{n+1}}$.

Hence, $\{1, 5, 5^2, \ldots, 5^{2^{n-1}-1}\} \equiv \{1, 5, 9, 13, \ldots, 2^{n+1}-3\} \pmod{2^{n+1}}$. Then

$$\prod_{k=0}^{2^{n-1}-1}(4k+1) \equiv \prod_{k=0}^{2^{n-1}-1} 5^k = 5^{2^{n-2}(2^{n-1}-1)} \equiv (1+2^n)^{2^{n-1}-1} \equiv 1 + 2^n \pmod{2^{n+1}}.$$

Thus,

$$\begin{aligned}(2^n+1)!! &= \prod_{k=0}^{2^{n-2}}(4k+1) \times \prod_{k=0}^{2^{n-2}-1}(4k+3) = \prod_{k=0}^{2^{n-2}}(4k+1) \times \prod_{k=2^{n-2}}^{2^{n-1}-1}(2^{n+1}-(4k+1)) \\ &\equiv (-1)^{2^{n-2}}(2^n+1)\prod_{k=0}^{2^{n-1}-1}(4k+1) \equiv (2^n+1)^2 \equiv 1 \pmod{2^{n+1}}.\end{aligned}$$

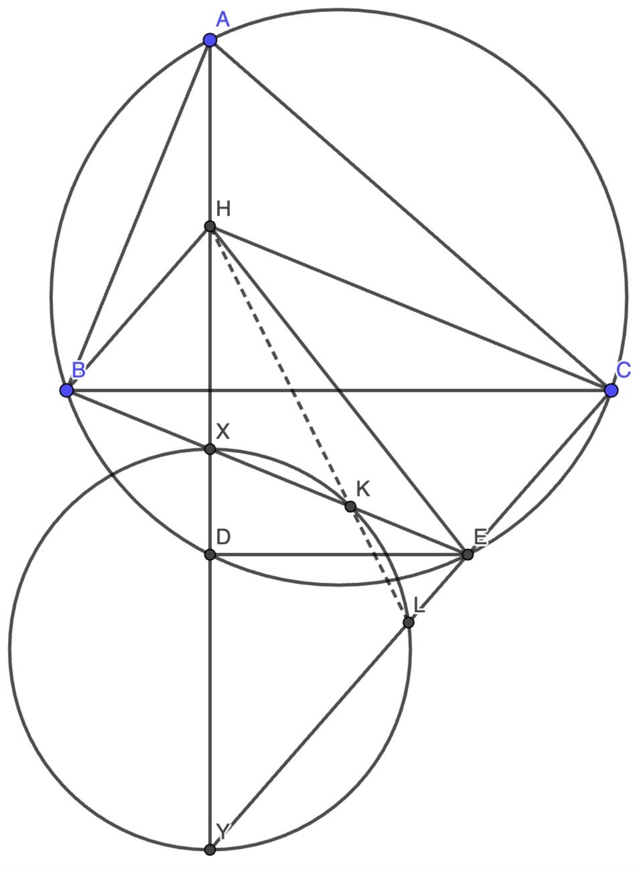
Hence, $\dfrac{(2^n+1)!!-1}{2^{n+1}}$ is an integer for all $n \geq 3$. Therefore, the answer is all integers $n \geq 3$. ■

3. Given triangle ABC with orthocenter H, the lines through points B and C that are perpendicular to lines AB and AC respectively, intersect line AH at points X and Y respectively. The circle with diameter XY intersects lines BX and CY a second time at points K and L respectively. Prove that points H, K, and L are collinear.

Solution. We use directed angles (mod $180°$). Let D be the second intersection of line AH and the circumcircle of triangle ABC, and let E be the intersection of lines BX and CY. Since $AB \perp BE$ and $AC \perp CE$, E must be the antipode of A with respect to triangle ABC, so $\measuredangle EDA = 90°$. Hence, $\measuredangle EDY = \measuredangle EDA = 90°$ and $\measuredangle EKY = \measuredangle XKY = 90°$, implying $EKDY$ is cyclic. Similarly, we get $\measuredangle XDE = \measuredangle XLE = 90°$, so $ELDX$ is cyclic.

Since lines BH and CY are perpendicular to line AC, we have $BH \parallel CY$. Similarly, lines CH and BX are perpendicular to line AB, so we have $CH \parallel BX$. Then $\measuredangle BHD = \measuredangle BHY = \measuredangle EYH = \measuredangle EYD = \measuredangle EKD = \measuredangle BKD$, so $BHKD$ is cyclic. Similarly, $\measuredangle CLD = \measuredangle ELD = \measuredangle EXD = \measuredangle CHD$, so $CHDL$ is also cyclic.

Then $\measuredangle KHD = \measuredangle KBD = \measuredangle EBD = \measuredangle ECD = \measuredangle LCD = \measuredangle LHD$, which implies H, K, and L are collinear. ■



4. Let n be a positive integer. For any $\mathcal{S} \subseteq \{1, 2, \cdots, n\}$, let $f(\mathcal{S})$ be the set containing all positive integers at most n that have an odd number of factors in $\mathcal{S}$. How many subsets of $\{1, 2, \cdots, n\}$ can be turned into $\{1\}$ after finitely many (possibly zero) applications of f?

Solution. The answer is $2^{\lceil \log_2(\log_2(n+1)) \rceil}$. To show this, we first have a lemma:

Lemma. f is a bijection.

Proof of Lemma: We instead show that f is surjective on the power set $\mathbb{P}(\{1, 2, \cdots, n\})$; since this is also the domain and range of f is , this shows that f^{-1} exists and is well-defined, i.e. f is a bijection. □

Let $\mathcal{S}'$ be a target set; we wish to show a set $\mathcal{S}$ exists so that only those integers in $\mathcal{S}'$ have an odd number of factors in $\mathcal{S}$.

Initially define $\mathcal{S} = \varnothing$. For each integer $1 \le i \le n$, in order, we do the following:

> If $i \in \mathcal{S}'$ and there are an even number of factors of i in $\mathcal{S}$, or if $i \notin \mathcal{S}'$ and there are an odd number of factors of i in $\mathcal{S}$, we add i to $\mathcal{S}$. Otherwise, do nothing.

Notice that this makes it so that the number of factors of i in $\mathcal{S}$ is of the right parity, but importantly, it does not affect the number of factors for any number less than $\mathcal{S}$. Thus, repeating this process for all i in increasing order, we can create the set $\mathcal{S}$ as defined above. □

Because the subsets of $\{1, 2, \cdots, n\}$ then form "cycles", it suffices to find the least positive integer x such that $f^x(\{1\}) = \{1\}$.

Define the function $g(\mathcal{S})$ on multisets of positive integers from 1 to n as follows: for each $k \in \mathcal{S}$, we put every multiple of k at most n in the set $g(\mathcal{S})$. So for example, if $n = 7$ and $\mathcal{S} = \{1, 2, 5, 6\}$ we have

$$g(\mathcal{S}) = \{1, 2, 2, 3, 4, 4, 5, 5, 6, 6, 6, 7\}.$$

Notice that the number of times an element appears in $g(\mathcal{S})$ is the same as the number of its divisors in $\mathcal{S}$. This means that the parity of the number of times an element appears is the same after applying either f or g; the number of times some k appears in $g^x(\{1\})$ is the number of sequences $d_1, d_2, \cdots, d_{x+1}$ such that $d_1 = 1$, $d_{x+1} = k$, and $d_i \mid d_{i+1}$ for all $1 \le i \le x$. Let this number be $c(x, k)$.

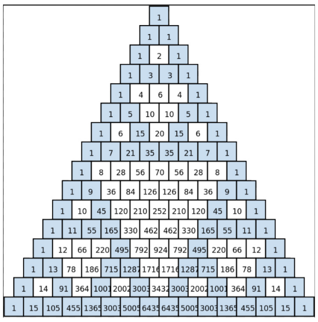
Let $b(x, k)$ denote the number of sequences $a_0, a_1, a_2, \cdots, a_x$ with $a_0 = 0$, $a_x = k$, and $a_0 \le a_1 \le \cdots \le a_k$. Notice that the exponents in the prime factorizations of the sequence d match with the possible sequences a. However, $b(x, k)$ has a closed form $\binom{x+k-1}{k}$ by stars-and-bars.

Thus, letting $x = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be the prime factorization, we have

$$c(x, k) = \prod_{i=1}^{t} b(x, k) = \prod_{i=1}^{t} \binom{x + k - 1}{k}.$$

Claim: The period of the parity of $\binom{x}{k}$ as x goes up is the smallest power of 2 greater than k.

Proof of Claim: Note that the set of odd numbers in the Pascal triangle form a Sierpinski triangle:



The period of the parity is the same as the period of the blue/white pattern going down and right a diagonal. This is the size of the length of the smallest "big triangle" that contains row k, which is the smallest power of 2 greater than k. $\square$

Because $c(x, k)$ is a product of a bunch of binomial coefficients, the period of the parity of $c(x, k)$ is the LCM of the periods of the parities of the binomial coefficients, and since they're all powers of two, it reduces to the largest one. That means that the period of f when starting at $\{1\}$ is after the largest of all the periods of $\binom{x}{k}$, as k ranges over the exponents of the primes in the prime factorizations of integers from 1 to n. This is just the smallest power of 2 greater than $\log_2 n$, which is $2^{\lceil \log_2(\log_2(n+1)) \rceil}$. $\blacksquare$

Remark: The answer for n is given by $a(n)$ in the OEIS sequence A063511.

Remark: This problem shares a similarity with the following puzzle:

> There are 100 lockers, numbered from 1 to 100. We first open every locker, then every second locker, then every third locker, and so on. Which lockers are left open?

In particular, this problem asks for $f(\{1, 2, \cdots, 100\})$. For a set S, if we only open every k^{th} locker if $k \in S$, the set of open lockers gives $f(S)$. The proof that f is surjective can be phrased like this: if we have a "goal" state of lockers, we can go from $k = 1$ to $k = 100$, choosing to open every kth locker only if the locker does not match our current goal.

However, knowing this does not help with finding the subsets associated with $\{1\}$.

Remark: It is possible to finish the second part (finding the period of $[\nu_2\left(\binom{n}{k}\right) == 0]$ with k constant) by appealing to Lucas's theorem. Indeed, the period is just the smallest power of two greater than k, which is true because the last $\lfloor \log_2 k \rfloor$ digits in the binary rep of n repeat after that many increments.

5. Find the largest positive integer k so that any binary string of length 2024 contains a palindromic substring of length at least k.

<u>*Solution.*</u> The answer is 4.

Lower bound: Consider any 4 digits sufficiently far from either side. If no two consecutive digits are equal, then a palindrome is formed. Otherwise, we have a substring of the form ?00? or ?11? (WLOG the first one). If both of the ?s are equal, then we are done. Otherwise, we have a substring of the form ?0001?. If the first digit is a 0, then the first four digits form a palindrome. Otherwise, the first five digits form a palindrome. Either way, we have found a palindrome with at least 4 digits.

Upper bound: Consider the string $110100110100110100\cdots$ of cycle 6 ad infinitum. It can be checked that none of the 5-substrings

$$11010, 10100, 01001, 10011, 00110, 01101$$

or the 6-substrings

$$110100, 101001, 010011, 100110, 001101, 011010$$

are palindromes. This means that no higher length palindromes exist either, because removing an equal amount from either side should give one of these substrings. ■

Remark: We can replace 2024 with any positive integer n. The maximum guaranteed length of a palindromic substring is 1 for $[1, 2]$, 2 for $[3, 4]$, 3 for $[5, 8]$ (consider the string 11101000), and 4 for $n \geq 9$.

6. For a real number x, let $\lfloor x \rfloor$ be the greatest integer not exceeding x. The sequence $\{a_n\}_{n\geq 1}$ of real numbers is defined as follows:

$$a_1 = 1, \text{ and } a_{n+1} = \frac{1}{2\lfloor a_n \rfloor - a_n + 1} \text{ for all } n \geq 1.$$

Find, with proof, the value of a_{2024}.

Solution.

First, note that if $a_k > 0$ for some integer $k \geq 1$, then

$$a_{k+1} = \frac{1}{2\lfloor a_k \rfloor - a_k + 1} > \frac{1}{2(a_k+1) - a_k + 1} = \frac{1}{a_k + 3} > 0.$$

Hence, a_n is positive for all integers $n \geq 1$.

We claim that for all $n \geq 2$, $a_n = a_{\frac{n-1}{2}} + 1$ if n is odd, and $a_n = \frac{a_{n/2}}{a_{n/2}+1}$ if n is even. We prove this by strong induction.

For the base case $n = 2$, note that $a_2 = \frac{1}{2\lfloor a_1 \rfloor - a_1 + 1} = \frac{1}{2} = \frac{a_1}{a_1+1}$, so the claim holds for this case.

Now, assume the claim holds for $n = 1, 2, \ldots, k$ for some integer $k \geq 2$. Then we will show the claim must hold for $n = k + 1$.

If k is even, then $k = 2\ell$ for some integer $\ell \geq 1$, and $a_k = \frac{a_\ell}{a_\ell + 1}$. We have

$$a_{k+1} = \frac{1}{2\lfloor a_k \rfloor - a_k + 1} = \frac{1}{2\left\lfloor \frac{a_\ell}{a_\ell+1} \right\rfloor - \frac{a_\ell}{a_\ell+1} + 1} = a_\ell + 1 = a_{\frac{(k+1)-1}{2}} + 1,$$

so the claim holds for $n = k + 1$ in this case.

If k is odd, then $k = 2\ell - 1$ for some integer $\ell \geq 2$, and $a_k = a_{\ell-1} + 1$. We have

$$a_{k+1} = \frac{1}{2\lfloor a_k \rfloor - a_k + 1} = \frac{1}{2\lfloor a_{\ell-1} \rfloor - a_{\ell-1} + 2} = \frac{\frac{1}{2\lfloor a_{\ell-1} \rfloor - a_{\ell-1} + 1}}{\frac{1}{2\lfloor a_{\ell-1} \rfloor - a_{\ell-1} + 1} + 1} = \frac{a_\ell}{a_\ell + 1} = \frac{a_{(k+1)/2}}{a_{(k+1)/2} + 1},$$

so the claim holds for $n = k + 1$ in this case.

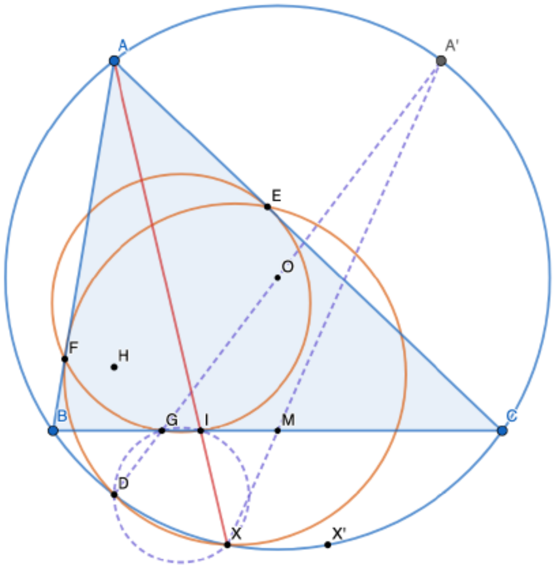
Therefore, the claim is proved for all integers $n \geq 2$. Note that $2024 = 11111101000_2$, so

$$a_3 = 2,\ a_7 = 3,\ a_{15} = 4,\ a_{31} = 5,\ a_{63} = 6,\ a_{126} = \frac{6}{7},\ a_{253} = \frac{13}{7},\ a_{506} = \frac{13}{20},\ a_{1012} = \frac{13}{33},$$

and lastly, $a_{2024} = \frac{13}{46}$. ■

7. Let ABC be an acute triangle with orthocenter H, circumcenter O, and circumcircle Ω. Points E and F are the feet of the altitudes from B to AC and C to AB, respectively. Let line AH intersect Ω again at point $D \neq A$. The circumcircle of DEF intersects Ω again at X, and AX intersects BC at I. The circumcircle of triangle IEF intersects BC again at G. If M is the midpoint of BC, prove that lines MX and OG intersect at a point on Ω.

<u>*Solution.*</u> Let the line through A parallel to BC intersect Ω again at A'. We claim that A' is the desired intersection point.



By radical axis on the circumcircles of ABC, $BCEF$, DEF, it follows that lines EF, BC, DX concur at some point P. Then by power of a point, we have

$$PI \cdot PG = PE \cdot PF = PB \cdot PC = PD \cdot PX,$$

so I, G, D, X are concyclic. Now taking perspectivity at D, it follows that

$$(A, X; B, C) \stackrel{D}{=} (XD \cap BC, AD \cap BC; B, C) = (EF \cap BC, AD \cap BC; B, C) = -1,$$

so $ABXC$ is harmonic. Now we show that points X, M, A' are collinear. Let $M' = AA' \cap BC$, and we will show that $M' = M$. Note that

$$(B, C; M', P_\infty^{BC}) \stackrel{X}{=} (B, C; A', X') = (C, B; A, X) = -1,$$

where X' is the reflection of X about OM, and the second step follows from the fact that cross-ratio is preserved under reflecting with respect to OM. It then follows that $M' = M$, so XMA' is a line.

Next, we show that OGA' is a line. We will prove the stronger statement that D, G, O, A' are collinear. It is clear that D, O, A' are collinear, so it suffices to prove that D, G, A' are collinear. This is because

$$\angle A'DX = \angle XDC + \angle CDA' = \angle XAC + \angle ACI = \angle XIC = \angle GDX,$$

so $OG \cap MX = A' \in \Omega$, as desired. ■

Remark: It is possible to prove that XMA' is a line without using projective geometry. To do this, let D' be the foot of the altitude from A to BC. By sine law, we have

$$\begin{aligned}\frac{BA}{BX}\Big/\frac{CA}{CX} &= \frac{\sin\angle ADB}{\sin\angle BDX}\Big/\frac{\sin\angle ADC}{\sin\angle CDX}\\ &= \frac{\sin\angle D'DB}{\sin\angle BDP}\Big/\frac{\sin\angle D'DC}{\sin\angle CDP}\\ &= \frac{D'B}{D'C}\Big/\frac{PB}{PC}\\ &= 1,\end{aligned}$$

so $BX \cdot CA = CX \cdot BA$, implying AX is the A-symmedian. In particular, if X' is the reflection of X about the perpendicular bisector of BC, AMX' is a line. Reflecting this line across the same perpendicular bisector yields $A'MX$ is a line, as desired.

8. Let $\varphi(n)$ denote the number of positive integers $m \leq n$ satisfying $\gcd(m, n) = 1$. Find all positive integers n for which $\varphi(\varphi(n))$ divides n.

<u>*Solution.*</u> The answer is $n = 1, n = 3, n = 2^a, n = 2^a 3^b, n = 2^a p$ for $p \in \{5, 7\}$ and $a, b \geq 1$. It is easy to verify that for each of these values of n, $\varphi(\varphi(n))$ indeed divides n. We have

n	$\varphi(\varphi(n))$
1, 2, 3	1
$2^a, a \geq 2$	2^{a-2}
$2^a 3$	2^{a-1}
$2^a 3^b, b \geq 2$	$2^a 3^{b-2}$
$2^a p, p \in \{5, 7\}$	2^a

It remains to show that these are the only ones.

First, we show that if $n > 3$, we must have n even. Instrumental is the following lemma:

Lemma 1: If n has k distinct odd prime factors, then $v_2(\varphi(n)) \geq v_2(n) + k - 1 \geq v_2(n) - 1$, where $v_2(n)$ denotes the power of 2 in the prime factorization of n. In particular, if n is not a power of 2, then $v_2(\varphi(n)) \geq v_2(n)$.

Proof of Lemma 1: Write $n = 2^a m$ for some odd m, so that $v_2(n) = a$. We have $\varphi(n) = \varphi(m)$ if $a = 0$ (i.e. n is odd) or $\varphi(n) = 2^{a-1}\varphi(m)$ otherwise. Then we have that for each odd prime $p \mid m$, $2 \mid p - 1 \mid \varphi(m)$; this means that $v_2(\varphi(m)) \geq k$. Hence $v_2(\varphi(n)) \geq a - 1 + v_2(\varphi(m)) \geq a - k - 1$ as desired. □

We now use Lemma 1 to prove our claim. Suppose $n > 3$ is odd. Then either n is divisible by at least two distinct primes or is a power of some odd prime. However, if n is divisible by at least two odd primes, say, p, q, then $4 \mid (p - 1)(q - 1) \mid \varphi(n)$, i.e., $v_2(\varphi(n)) \geq 2$. Then from our lemma $v_2(\varphi(\varphi(n))) \geq 1$, i.e., $\varphi(\varphi(n))$ is even and thus cannot divide n.

Now that we've shown that n is even, we show that n is divisible by at most one odd prime. We use the following lemma:

Lemma 2: If $\varphi(n) = 2^d$ for some positive integer d, then we must have n is a product of a power of 2 and distinct Fermat primes, i.e., primes of the form $2^k + 1$.

Proof of Lemma 2: if an odd prime q divides n, then $\varphi(q) = q - 1$ divides $\varphi(n) = 2^d$. This implies that $q = 2^k + 1$ for some positive integer k. Moreover, if $q^2 \mid n$, then $q \mid \varphi(n)$, which cannot be; this means that $v_q(n) \leq 1$, and any Fermat prime can appear at most once in the factorization of n. □

Write $v_2(n) = a$. If n is divisible by at least two odd primes, say, p, q then $\varphi(pq) \mid \varphi(n)$ and $v_2(\varphi(n)) = v_2(n) - 1 + v_2(\varphi(pq)) \geq v_2(\varphi(n)) + 1$. On one hand, if $\varphi(pq)$ is not a power of 2, we write $\varphi(pq) = 2^c m$ for some odd $m > 1$. We note that $\varphi(pq) = (p - 1)(q - 1)$ which is the product of two even numbers and thus divisible by 4; we then have $c = v_2(\varphi(pq)) \geq 2$, and so $2^{a+1}m$ divides $\varphi(n)$. Then, we get in turn $\varphi(2^{a+1})\varphi(m) = 2^a\varphi(m)$ must divide $\varphi(\varphi(n))$. However, since m is odd and $m > 1$, $\varphi(m)$ is even, and so $v_2(\varphi(m) \geq 1$; this means that 2^{a+1} divides $\varphi(\varphi(n))$, which then precludes the possibility of $\varphi(\varphi(n))$ dividing n. On the other hand, if $\varphi(pq)$ is a power of 2, we must have that in fact $p \geq 3 = 2 + 1$ and $q \geq 5 = 2^2 + 1$, so $v_2(\varphi(pq)) \geq 3$ and so $v_2(\varphi(n)) \geq v_2(n) + 2$. From Lemma 1 we then have $v_2(\varphi(\varphi(n))) \geq v_2(n) + 1$, and so $\varphi(\varphi(n))$ cannot divide n.

Thus, we have either $n = 2^a$, which indeed satisfies the condition, or $n = 2^a p^b$ for some odd prime p. Now, if $p = 3$, then we also have that n satisfies the condition. Thus, we look at all primes $p > 3$; we show that $p \in \{5, 7\}$ and $b = 1$.

First, suppose $p > 7$. Then either $\varphi(p) = 2^c$ for some $c \geq 3$, or $\varphi(p) = 2^c m$ for some $c \geq 1$ and odd $m \geq 5$. In the former case, we then have $v_2(\varphi(n)) \geq v_2(n) - 1 + c \geq v_2(n) + 2$. Thus, $v_2(\varphi(\varphi(n))) \geq v_2(n) + 1 > v_2(n)$, and so $\varphi(\varphi(n))$ cannot divide n. In the latter case, we write $\varphi(n) = 2^{a+c-1}m$, so $\varphi(\varphi(n)) = 2^{a+c-2}\varphi(m)$. If $\varphi(m)$ has at least one odd prime factor q, then $q < p$ so $q \nmid n$, but $q \mid \varphi(\varphi(n))$. Thus n cannot satisfy the condition. We must then have $\varphi(m) = 2^d$ for some d, and so by Lemma 2 and the assumption that m is odd we have that m is a product of distinct Fermat primes. Then the condition $m \geq 5$ tells us that there exists at least one odd $q = 2^k + 1$ with $k \geq 2$ dividing m, and so $4 \mid 2^k \mid \varphi(m)$. This means that $v_2(\varphi(\varphi(n))) \geq a + c > a$, so $\varphi(\varphi(n))$ cannot divide n.

To finish off the proof, we look at $n = 2^a p^b$. We know that for $b = 1$, n satisfies the desired condition. However, if $n = 2^a 5^b$ for $b \geq 2$, then $\varphi(\varphi(n)) = 2^{a+2}5^{b-2}$ which does not divide n; if $n = 2^a 7^b$ for $b \geq 2$, we have $\varphi(\varphi(n)) = 2^{a+1}7^{b-2}3$ which also cannot divide n. Thus, the only n that satisfy the property are given in the table above. ■